MATH 210, MANIFOLDS III, Spring 2006

Homework Assignment I: Tubular neighborhoods and approximations of continuous maps by smooth maps, due Monday 4/17-2006

1. Fill in the details of the proof of the tubular neighborhood theorem for a closed submanifold M of $\mathbb{R}^n$. Namely, let V_ϵ be the collection of all vectors normal to M and having length less than $\epsilon > 0$, i.e.,

$$V_\epsilon = \{(p, \xi) \mid p \in M,\ \xi \in (T_pM)^\perp \text{ and } \|\xi\| < \epsilon\}.$$

Define $\varphi\colon V_\epsilon \to \mathbb{R}^n$ by $\varphi(p, \xi) = p + \xi$. The tubular neighborhood theorem asserts that φ is a diffeomorphism on its image if $\epsilon > 0$ is small enough.

(a) Identify $T_{(p,0)}V_\epsilon$ with $T_p\mathbb{R}^n$ for $p \in M$ and show that $D_{(p,0)}\varphi = id$. Conclude from this that φ is an immersion, when $\epsilon > 0$ is sufficiently small.

(b) Prove that φ is one-to-one if $\epsilon > 0$ is small enough. (Use, for instance, the fact that $\varphi|_M = id$.)

2. Let M be closed and $m = \dim M < n$. Prove that every continuous map $f\colon M \to S^n$ is homotopic to a map sending M to one point.

3. Let M be a manifold of dimension n. Assume for the sake of simplicity that M is closed. Prove that there exists a continuous map $f\colon S^1 \to M$ whose image is the entire manifold M. (Remark: Such a map f cannot be C^1 when $n > 1$, by Sard's theorem.)

4. Use Problem 2 to show that S^m is not homeomorphic to S^n whenever $n > m$.

5. Let M^m be a manifold, which we assume to be closed for the sake of simplicity. The collection of immersions of M into $\mathbb{R}^n$ is dense in $C^\infty(M, \mathbb{R}^n)$ if $n \geq 2m$. Either prove this fact for C^0-topology or read its proof in Lee's book (Theorem 10.8) or elsewhere (e.g., in Hirsch's book). Likewise, embeddings are dense in $C^\infty(M, \mathbb{R}^n)$ if $n \geq 2m+1$. The same holds with $\mathbb{R}^n$ replaced by an arbitrary manifold N of dimension $n \geq 2m$ or, respectively, $n \geq 2m + 1$ and the proofs are only marginally more difficult. Moreover, when M is closed as assumed above, immersions or embeddings form open dense sets in $C^\infty(M, N)$.