MATH 210, MANIFOLDS III, Spring 2006

Final Homework Assignment
due Wednesday 05/31-2006

1. Let M^n and N^n be closed, oriented manifolds and let $f\colon M \to N$ be a map of non-zero degree. Prove that $f^*\colon H^*(N) \to H^*(M)$ is a monomorphism.

Hint. Use the Poincaré duality. Let $\omega \in H^n(N)$ be a non-zero element. Then $f^*\omega \neq 0$, for $\deg f \neq 0$. For any non-zero $\alpha \in H^*(N)$, there exists $\beta \in H^*(N)$ such that $\alpha \cup \beta = \omega$. Derive from this that $f^*\alpha \neq 0$.

2. Let M be an oriented, compact manifold, possibly with boundary. Let v be a vector field, pointing outward at the boundary points of M. Assuming that the zeros of v are non-degenerate (this is a generic condition), define the Euler characteristic of M by the same formula as for manifolds without boundary: $\chi(M) = \sum \sigma_p(v)$. Here the summation extends over all zeros p of v and the index $\sigma_p(v)$ is defined in the same way as for closed manifolds.

(a) Prove that $\chi(M)$ is well defined, i.e., independent of v.
(b) Assume that M is odd-dimensional. Prove that $\chi(\partial M) = 2\chi(M)$.
(c) Prove that $\mathbb{C}P^{2n}$ does not bound a compact, oriented manifold.
(d) Prove that $\mathbb{R}P^{2n}$ does not bound a compact manifold.

Hints and remarks. Proving (b), one may argue as follows. Consider the double N of M, with ∂M canonically embedded into N. Utilizing the tubular neighborhood theorem, modify v near ∂M and then extend it to a vector field w on N such that w is tangent to ∂M. Then counting zeros of w and using fact that $\chi(N) = 0$ conclude that $\chi(\partial M) = 2\chi(M)$. To establish (d), first observe that orientations are not really essential in (a) and (b). It should be clear that, if nothing else, (a) and (b) hold without orientations for $\chi \mod 2$.

3*. Prove that every continuous map $f\colon \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point. Show that there exists a diffeomorphism $f\colon \mathbb{R}P^{2n+1} \to \mathbb{R}P^{2n+1}$ without fixed points.

Hint. Here you may, for instance, first calculate the de Rham cohomology $H^*(\mathbb{R}P^{2n})$ and then use the Lefschetz fixed point theorem.

4. Let M^m and N^n be disjoint, closed, oriented submanifolds of $\mathbb{R}^{k+1}$ such that $m + n = k$. Recall from Homework Assignment IV that he linking number $Lk(M, N)$ is the degree of the map $F\colon M \times N \to S^k$ defined by $F(x, y) = (x - y)/\|x - y\|$.

(a) Assume that $N = S^{k-n}$ is a small sphere in a fiber of a tubular neighborhood of M, centered at the origin in the fiber. Prove that $Lk(M, N) = 1$, when N is suitably oriented.

(b) Assume that $N = \partial X$, where X is a compact oriented submanifold with boundary in $\mathbb{R}^{k+1}$. Since $N = \partial X$ is disjoint from M, the intersection number $M \cdot X$ is defined. Prove that $Lk(M, N) = M \cdot X$.

Hints and remarks. In (b), you may assume without loss of generality that X is transverse to M. Let $\{p_1, \ldots, p_l\} = M \cap X$. Let $D_1, \ldots, D_l$ be small balls in X centered at $p_1, \ldots, p_l$. Then the map F extends to a map $\Phi\colon M \times Y$, where Y is the complement of $D_1, \ldots, D_l$ in X. It follows that the $\deg F = \sum \deg \Phi|_{M \times S_i}$, where $S_i = \partial D_i$, or in other words $Lk(M, N) = \sum Lk(M, S_i)$. Furthermore, taking orientations into account, show that $\sum Lk(M, S_i) = M \cdot X$. Note that the embedding $X \hookrightarrow \mathbb{R}^{k+1}$ can be replaced by an arbitrary map $f\colon X \to \mathbb{R}^{k+1}$, as long as $f|_{\partial X}$ is an orientation preserving diffeomorphism from ∂X to N.

5. Let E be an oriented vector bundle over a closed, oriented manifold M. Let s be a section of E transverse to the zero section. Then $L = \{s = 0\} = s(M) \cap M$ is a smooth, oriented submanifold of M of dimension $k = \dim M - \operatorname{rk} E$.

(a) Prove that the normal bundle to L in M is isomorphic to $E|_L$.

(b) Prove that the Euler class $e(E)$ is Poincaré dual to L, i.e., $\int_L \alpha = \int_M e(E) \wedge \alpha$, for every closed k-form α. (Here we use the notation $e(E)$ simultaneously for the cohomology class $e(E)$ and for a differential form representing this class.)