MATH 209, MANIFOLDS II, WINTER 2006

Homework Assignment VI: Orientations and integration, due Tuesday 2/28–2006

1. Let $F\colon S^n \to S^n$ be the antipodal map.

(a) Prove that F is orientation preserving when n is odd and orientation reversing when n is even.

(b) Prove that $\mathbb{R}P^n$ is orientable if and only if n is odd.

2. Let $F\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Show that F is necessarily orientation preserving at its regular points, i.e., $F^* dx \wedge dy = f\, dx \wedge dy$ with $f \geq 0$.

3. Let N be a hypersurface in M and let ω be a volume form on M.

(a) Let v be a vector field nowhere tangent to N. Prove that $i_v\omega|_N$ is a volume form on N.

(b) Prove that $i_v\omega|_N = i_w\omega|_N$ if $v - w$ is tangent to N.

Remark. Assume that the hypersurface N is the boundary of M. Then the construction of Part (a) gives an alternative description of the orientation induced on N. Indeed, let v point outward and let an orientation of M be determined by ω. Then the induced orientation of $N = \partial M$ is determined by $i_v\omega$ and is well defined. Note also that in both (a) and (b) it suffices to have v defined only along N.

4. Let $M \subset \mathbb{R}^3$ be the graph of a function $z = f(x, y)$ with (x, y) lying in some bounded closed domain U of $\mathbb{R}^2$. Let $\mathbf{v}$ be a vector field in $\mathbb{R}^3$ defined on a neighborhood of M. Recall from vector calculus that the surface integral of $\mathbf{v}$ over M is defined as

$$\iint_M \mathbf{v} \cdot d\mathbf{S} = \iint_M \mathbf{v} \cdot \mathbf{n}\, dS = \iint_U (\mathbf{v} \cdot \mathbf{n})(x, y)\sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}\, dx\, dy,$$

where $\mathbf{n}$ is the unit upward normal vector field to M. Let $\omega = dx \wedge dy \wedge dz$.

(a) Prove that $\iint_M \mathbf{v} \cdot d\mathbf{S} = \int_M i_{\mathbf{v}}\omega$, where the orientation of M is induced by ω and $\mathbf{n}$ as in Problem 3.

(b) Prove that $F^* i_{\mathbf{n}}\omega = \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}\, dx \wedge dy$, where $F\colon U \to M$ is the natural diffeomorphism $(x, y) \mapsto (x, y, f(x, y))$.

Remark. Since every hypersurface is locally a graph (in some orthogonal coordinates), this statement indicates that in general the integral of $\mathbf{v}$ over a hypersurface in the sense of vector calculus is equal to the integral of $i_{\mathbf{v}}\omega$.

5. Let $\omega \in \Omega^2(M)$ and $u\colon [0, 1] \times [0, 1] \to M$ be a smooth map. Prove that $u^*\omega = \omega\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s}\right) dt \wedge ds$ and, as a consequence,

$$\int_u \omega = \int_0^1 \int_0^1 \omega\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s}\right) dt\, ds.$$

6. Let $\omega = x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy$. Evaluate $\int_{S^2_R} \omega$, where S^2_R is the sphere of radius R centered at the origin.