MATH 209, MANIFOLDS II, WINTER 2006

Homework Assignment V: More on Differential Forms, due Tuesday 2/14–2006

Throughout this assignment we assume that M a smooth manifold of dimension n.

1. Prove, using the original definition, that for a vector field v on M and $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$, we have $L_v(\alpha \wedge \beta) = (L_v\alpha) \wedge \beta + \alpha \wedge (L_v\beta)$.

2. Let $\omega = dx_1 \wedge \cdots \wedge dx_n$ on $\mathbb{R}^n$. Prove that for any vector field $v = (v_1, \ldots, v_n)$ on $\mathbb{R}^n$, we have $L_v\omega = (div\, v)\omega$, where $div\, v := (\partial v_1/\partial x_1) + \cdots + (\partial v_n/\partial x_n)$.

Remark. By definition, a volume form on M is an n-form ω such that $\omega_x \neq 0$ for any $x \in M$, i.e., ω is a non-vanishing n-form. Define the divergence $div_\omega\, v$ of v with respect to ω by $L_v\omega = (div_\omega\, v)\omega$. Then $div_\omega\, v$ measures to what extent the flow φ_t of v is volume expanding or contracting. In particular, $div_\omega\, v = 0$ is equivalent to $\varphi_t^*\omega = \omega$, i.e., the flow is volume preserving.

3. Recall that a distribution $\mathcal{D}$ on M is a smooth sub-bundle of TM, i.e., a family of subspaces $\mathcal{D}_x \subset T_xM$ depending smoothly on x. A distribution is said to be involutive or integrable if for any two vector fields v and w tangent to $\mathcal{D}$ (i.e., such that $v(x)$ and $w(x)$ are in $\mathcal{D}_x$ for all $x \in M$), the Lie bracket $[v, w]$ is again tangent to $\mathcal{D}$. (See the textbook for more details.)

(a) Is the distribution $\mathcal{D}$ spanned by $v = \partial_x$ and $w = x\partial_z - \partial_y$ on $\mathbb{R}^3$ integrable? Conclude from your solution that every function which is constant along the distribution $\mathcal{D}$ (i.e., such that $L_vf = L_wf = 0$ everywhere) must be constant. Sketch this distribution.

(b) Let $f\colon M \to \mathbb{R}$ be a function without critical points on M, i.e., such that $df_x \neq 0$ for any $x \in M$. Show that the distribution $\mathcal{D}_x := \ker df_x$ is integrable. (What are integral submanifolds for this distribution?)

4. Let α be a non-vanishing one-form on M. Consider the distribution $\mathcal{D}_x = \ker \alpha_x$. (For instance, the distribution from Problem 3(a) can be given as $\ker(dz + x\, dy)$.) Prove that $\mathcal{D}$ is involutive if and only if $\alpha \wedge d\alpha = 0$.

Hint. First prove that $\alpha \wedge d\alpha = 0$ is equivalent to that $d\alpha$ vanishes on $\mathcal{D}$, i.e., $d\alpha(v, w) =$ for any vectors v and w tangent to $\mathcal{D}$. Then use the expression $d\alpha(v, w) = L_v\alpha(w) - L_w\alpha(v) - \alpha([v, w])$. The result of Problem 4 is a form of the Frobenius theorem for distributions of codimension one. What examples of involutive distributions of codimension one do you know? Try to construct such a distribution (or rather a foliation) on S^3.

5*. Let φ^t and ψ^t be, respectively, the flows of vector fields v and w on M. Fix $p \in M$ and set $\gamma(t) = \psi^{-t}\varphi^{-t}\psi^t\varphi^t(p)$. Prove that $\gamma'(0) = 0$. Then show that $\gamma''(0)$ is well defined and equal to $2[v, w](p)$.

Remark. This explicitly shows that $[v, w]$ measures to what extent the flows of v and w do not commute. The difficult part is the identity $\gamma''(0) = 2[v, w](p)$. You may want to first look into this identity for linear vector fields on $\mathbb{R}^n$, before dealing with the general case. (See also Spivak's *A Comprehensive Introduction to Differential Geometry*, vol. I, p. 159–162.)