

MATH 209, MANIFOLDS II, WINTER 2006

Homework Assignment II: Vector bundles,  
due Tuesday 1/24-2006

1. Let  $E_{\mathbb{C}}$  be a complex line bundle and let  $E_{\mathbb{R}}$  be the same line bundle regarded as a real vector bundle. (Thus  $\text{rk}_{\mathbb{R}} E_{\mathbb{R}} = 2$ .) Prove that  $E_{\mathbb{C}}$  is trivial if and only if  $E_{\mathbb{R}}$  is trivial.

2. Let  $E$  be a real vector bundle over a compact manifold  $M$ . Assume that  $\text{rk } E > \dim M$ . Prove that  $E$  has a non-vanishing section.

Remark. Hint: use the transversality theorem. Even if you cannot quite prove the assertion, try to come up with a plausible reasoning and indicate what it is exactly that you are not proving.

3. Problem 5-3 (Chapter 5) on page 121 of the textbook.

Remark. You may also want to take a look at Problems 5-4 and 5-5. The first of these is done in class, skipping details which are a bit tedious. In any event, 5-3 – 5-5 will give you a better understanding of how transition functions work.

4. Problems 5-2, 5-6, and 5-12 (Chapter 5) on pages 121 and 122 of the textbook.

Remark. Note that a smooth vector bundle which is trivial as a smooth vector bundle is also trivial as a continuous vector bundle. Then in 5-2, it suffices to show that  $E$  is not trivial as a continuous vector bundle.

5. Prove that the tautological line bundle  $E$  over  $\mathbb{R}P^n = \text{Gr}_{\mathbb{R}}(1, n+1)$  is non-trivial. What is the unit sphere bundle  $SE$  in this case? Identify  $SE$  and the projection  $SE \rightarrow \mathbb{R}P^n$ .

Hint. First observe that the restriction of a trivial vector bundle to any subset (submanifold) is necessarily trivial. Then show that the restriction of  $E$  to  $\text{Gr}_{\mathbb{R}}(1, 2)$  is the tautological line bundle over  $\text{Gr}_{\mathbb{R}}(1, 2)$ , which is non-trivial by Problem 4.

6. Let  $E$  be the tautological complex line bundle over  $\mathbb{C}P^n = \text{Gr}_{\mathbb{C}}(1, n+1)$ .

- (a) Prove that  $SE$  is diffeomorphic to  $S^{2n+1}$ .
- (b) Prove furthermore that the natural projection  $SE \rightarrow \mathbb{C}P^n$  is (what does this mean?) the Hopf bundle  $S^{2n+1} \rightarrow \mathbb{C}P^n$ .
- (c) Sketch the fibers of the Hopf bundle  $S^3 \rightarrow \mathbb{C}P^1$  in  $\mathbb{R}^3 = S^3 \setminus \{\text{one point}\}$ .

Remark. This is an important problem which relies on some definitions that you may need to look up or ask me about, e.g., the definition of the Hopf bundle, although I have discussed some of them in class. Note the similarity with Problem 5. Also,  $\mathbb{C}P^1$  is diffeomorphic to  $S^2$ . Why? Define a homeomorphism if not a diffeomorphism. If we accept the fact that  $S^3$  is not homeomorphic to  $S^2 \times S^1$  or more generally that  $S^{2n+1}$  is not homeomorphic to  $\mathbb{C}P^n \times S^1$  (e.g., because  $\pi_1(S^{2n+1}) = 0$  while  $\pi_1(\mathbb{C}P^n \times S^1) = \mathbb{Z}$ ), we conclude that  $E$  is non-trivial.