MATH 209, MANIFOLDS II, WINTER 2006

Final
due Thursday 3/16–2006

Throughout the exam all manifolds, maps, and homotopies are assumed to be smooth.

1. Let N be a compact oriented n-dimensional submanifold with connected boundary in a manifold M. Prove that the restriction map $H^{n-1}(M) \to H^{n-1}(\partial N)$ is zero.

2. Let U be an open subset with in a manifold M. Assume that $H^{k-1}(M \setminus \bar{U}) = 0$ for some $k \geq 1$, where $\bar{U}$ is the closure of U. Let α be an exact k-form on M supported in U. Prove that for any open set $V \supset \bar{U}$, there exists a primitive of α supported in V.

Hint: argue as in the proof of Lemma 15.22 of the textbook. Remark: The assertion of the problem (combined with the Poincaré lemma, homotopy invariance of de Rham cohomology, and the calculation of $H^*(S^n)$) readily implies the Poincaré lemma for forms with compact support.

3. Problem 15-7 on page 408 of the textbook. (For the result to hold as stated one should also require M_1 and M_2 to be closed.) Prove, as a consequence, that $H^1(\Sigma_g) = \mathbb{R}^{2g}$, where Σ_g is the sphere with g handles.

4. For any $k \in \mathbb{Z}$ and $n > 0$ construct a smooth map $f\colon S^n \to S^n$ with $\deg f = k$.

Remark. We will show next quarter that two maps f_1 and f_0 from S^n to itself are homotopic if and only if $\deg f_0 = \deg f_1$.

5. Let M be a compact n-dimensional manifold with non-empty boundary. Prove that $H^n(M) = 0$.

Remark. More generally, $H^n_c(M) = 0$ once $\partial M \neq \emptyset$ even when M is not compact.

6 (Hopf invariant). The goal of this problem is to introduce a homotopy invariant of maps $f\colon S^3 \to S^2$, the so-called Hopf invariant, and to establish some of its properties.

(a) Let $\omega \in \Omega^2(S^2)$ be an arbitrary form with $\int_{S^2} \omega = 1$. Since $H^2(S^3) = 0$, the pull back $f^*\omega$ is exact. Define the Hopf invariant of f by $H(f) = \int_{S^3} \lambda \wedge f^*\omega$, where λ is a primitive of $f^*\omega$, i.e., $d\lambda = f^*\omega$. Prove that $H(f)$ is well-defined, i.e., independent of the choice of λ when ω is fixed and, furthermore, independent of the choice of ω.

(b) Evaluate the Hopf invariant for the Hopf fibration $S^3 \to S^2$.

(c) Prove that $H(f)$ is homotopy invariant.

(d) Consider maps $h\colon S^3 \to S^3$ and $g\colon S^2 \to S^2$. Prove that $H(g \circ f) = \deg(g)^2 H(f)$ and $H(f \circ h) = \deg(h) H(f)$.

Remarks. Next quarter we will show that $H(f) \in \mathbb{Z}$ for all f. It follows from Parts (b) and (d), and Problem 2, that $H(f)$ can assume arbitrary integer values. Furthermore, (b) and (c) imply that the Hopf fibration map is not homotopic to identity. The converse of (c) also holds: two maps f_1 and f_0 from S^3 to S^2 are homotopic if and only if $H(f_0) = H(f_1)$.

7. Prove that a real vector bundle over S^1 is trivial if and only if it is orientable.