MATH 209, MANIFOLDS II, WINTER 2024

Homework Assignment V: Orientations and integration

1. Let $F\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Show that F is necessarily orientation preserving at its regular points, i.e., $F^*dx \wedge dy = f\, dx \wedge dy$ with $f \geq 0$.

2. Let N be a hypersurface in M and let ω be a volume form on M, i.e., a non-vanishing top-degree form.

(a) Let v be a vector field nowhere tangent to N. Prove that $i_v\omega|_N$ is a volume form on N.

(b) Prove that $i_v\omega|_N = i_w\omega|_N$ if $v - w$ is tangent to N.

Remark. Assume that the hypersurface N is the boundary of M. Then the construction of Part (a) gives an alternative description of the orientation induced on N. Indeed, let v point outward and let an orientation of M be determined by ω. Then the induced orientation of $N = \partial M$ is determined by $i_v\omega$ and is well defined. Note also that in both (a) and (b) it suffices to have v defined only along N.

3. Let $\omega \in \Omega^2(M)$ and $u\colon D \to M$ be a smooth map, where D is a compact domain in $\mathbb{R}^2$ with smooth boundary. Prove that in the notation introduced in class

$$u^*\omega = \omega\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s}\right) dt \wedge ds$$

where t and s are the coordinates on the domain D of u and, as a consequence,

$$\int_u \omega = \iint_D \omega\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s}\right) dt\, ds.$$

4. In the setting of the previous problem with $M = \mathbb{R}^3$, let us think of $u = (u_1, u_2, u_3)\colon D \to \mathbb{R}^3$ as a parametrization of a surface by $D \subset \mathbb{R}^2$ in the sense of vector calculus. We assume for the sake of simplicity that $u = u(D)$ is an embedded surface with boundary with the orientation inherited from that of D. Then

$$\frac{\partial u}{\partial t} = \left(\frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t}, \frac{\partial u_3}{\partial t}\right) \text{ and } \frac{\partial u}{\partial s} = \left(\frac{\partial u_1}{\partial s}, \frac{\partial u_2}{\partial s}, \frac{\partial u_3}{\partial s}\right).$$

- Prove that the pull-back by u of the Riemannian area form on S coming from the metric induced by the inner product on $\mathbb{R}^3$ is
$$\left\|\frac{\partial u}{\partial t} \times \frac{\partial u}{\partial s}\right\| dt \wedge ds.$$
- Let $\vec{v}$ be a vector field on $\mathbb{R}^3$ and $\omega = i_{\vec{v}}(dx \wedge dy \wedge dz)$. Prove that
$$u^*\omega = \vec{v} \cdot \left(\frac{\partial u}{\partial t} \times \frac{\partial u}{\partial s}\right) dt \wedge ds,$$
and hence, in the sense of the standard vector calculus,
$$\int_u \omega = \iint_S \vec{v} \cdot dS.$$