MATH 209, MANIFOLDS II, WINTER 2024

Homework Assignment IV: Differential Forms

Throughout this assignment we assume that M a smooth manifold of dimension n.

1. Prove that for a vector field v on M and $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$, we have $i_v(\alpha\wedge\beta) = (i_v\alpha)\wedge\beta + (-1)^k\alpha\wedge(i_v\beta)$.

Remark. This is a consequence of a similar identity in linear algebra: $i_v(\alpha\wedge\beta) = (i_v\alpha)\wedge\beta + (-1)^k\alpha\wedge(i_v\beta)$, where $v \in V$ and $\alpha \in \bigwedge^k V^*$ and $\beta \in \bigwedge^l V^*$.

2. Prove that every k-form ω on M can be written as a (locally finite) sum of products of compactly supported one-forms. More precisely, there exist compactly supported one-forms α_i^j with i ranging within some countable set and $j = 1, \ldots, k$ such that the sets $F_i = \cup_j \mathit{supp}\,(\alpha_i^j)$ form a locally finite cover of M and

$$\omega = \sum_i \alpha_i^1 \wedge \cdots \wedge \alpha_i^k.$$

Hint. Here is one possible approach. Using a partition of unity associated with a locally finite cover by coordinate charts reduce the question to the case where ω is supported in a coordinate chart U with coordinates $(x_1, \ldots, x_n)$. Then, use a cut-off function equal to one on $\mathit{supp}\,(\omega)$ and vanishing outside of U to extend each dx_l to a smooth one form β_l on M equal to dx_l near $\mathit{supp}\,\omega$ and vanishing outside of U.

3. Let $F\colon M \to N$ be a smooth map. As we have seen, it induces the pull-back map of algebras $F^*\colon \Omega^*(N) \to \Omega^*(M)$. Prove that $dF^* = F^*d$.

Hint. Observe that, utilizing the properties of d and F and Problem 2, it is enough to prove this for functions and one-forms.

4. Problem 14-6 (page 375) from Chapter 14 of the textbook.

5. The goal of this problem is to show that grad, curl, and div are just particular cases of the de Rham differential on $M = \mathbb{R}^3$. Let us equip $\mathbb{R}^3$ with the standard inner product $\langle\cdot\,,\cdot\rangle$ and denote by $\mathcal{X}$ the space of smooth vector fields on $\mathbb{R}^3$. Define:

- $\Psi_1\colon \mathcal{X} \to \Omega^1(\mathbb{R}^3)$ by $\Psi_1(v) = \langle v, \cdot\rangle$ or, more explicitly, $\Psi_1(a\partial_x + b\partial_y + c\partial_z) = a\,dx + b\,dy + c\,dz$, where $\partial_x := \partial/\partial x$, etc.
- $\Psi_2\colon \mathcal{X} \to \Omega^2(\mathbb{R}^3)$ by $\Psi_2(v) = i_v(dx\wedge dy\wedge dz)$. (Work out an explicit expression for Ψ_2.)
- $\Psi_3\colon C^\infty(\mathbb{R}^3) \to \Omega^3(\mathbb{R}^3)$ by $\Psi_3(f) = f\,dx\wedge dy\wedge dz$.

Prove that the following diagram commutes (up to signs):

$$\begin{array}{ccccccc}
C^\infty(\mathbb{R}^3) & \xrightarrow{\mathit{grad}} & \mathcal{X} & \xrightarrow{\mathit{curl}} & \mathcal{X} & \xrightarrow{\mathit{div}} & C^\infty(\mathbb{R}^3) \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \Psi_1} & & \downarrow{\scriptstyle \Psi_2} & & \downarrow{\scriptstyle \Psi_3} \\
C^\infty(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3)
\end{array}$$

Remark. The first and the last square of the diagram make sense (how?) and commute for $\mathbb{R}^n$, but the middle one does not. Note also that in $\mathbb{R}^3$ the identities $\mathit{curl}\circ\mathit{grad} = 0$ and $\mathit{div}\circ\mathit{curl} = 0$ together are equivalent to $d^2 = 0$.