MATH 209, MANIFOLDS II, WINTER 2024

Final
due Thursday, 03/14, in class

Throughout the exam all manifolds, maps, and homotopies are assumed to be smooth.

1. Which of the following manifolds are orientable and which are not:
(a) $\mathbb{R}P^2$, $\mathbb{R}P^3$, the special orthogonal group SO(3);
(b) the complex projective space $\mathbb{C}P^n$.
Justify your answer in some detail.

2. Prove that a real vector bundle over S^1 is trivial if and only if it is orientable. Remark: You can use without proof the fact that a vector bundle $E \to M$, where M is a compact manifold, has a non-vanishing section when $\operatorname{rk} E > \dim M$; see Problem 3 in Homework 2.

3. Let D be a closed two-dimensional disk. Show that there is no smooth map $f\colon D \to \partial D$ such that $f\,|_{\partial D} = id$.

4. Let A be the surface area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ with respect to the induced metric. Prove that $A = nV$, where V be the volume of the unit ball in $\mathbb{R}^n$. Hint: show that the Riemannian volume form on S^{n-1} (or the area form if you wish to distinguish it from the form on $\mathbb{R}^n$) is the restriction to S^{n-1} of the form

$$\sigma = \sum_{j=1}^{n} (-1)^{j-1} x_j \, dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n,$$

defined on $\mathbb{R}^n$, and then use Stokes' theorem.

5. Consider the standard symplectic form

$$\omega = dp_1 \wedge dq_1 + \ldots + dp_n \wedge dq_n$$

on $\mathbb{R}^{2n}$ with coordinates $(p_1, q_1, \ldots, p_n, q_n)$. Let us identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by setting $z_k = p_k + iq_k$. Let A be a linear unitary transformation of $\mathbb{C}^n$. Show that A is symplectic, i.e., $A^*\omega = \omega$. Give an example of a symplectic linear transformation which is not unitary.

6. Let $F\colon P \to M$ be a surjective (i.e., onto) smooth map and let α be a differential form on M. Show that $\alpha = 0$ whenever $F^*\alpha = 0$. (The converse is obvious.) Hint: you need to use Sard's lemma.

7*. Poincaré's lemma asserts that every closed differential form α on $\mathbb{R}^n$ is exact: $d\alpha = 0$ iff $\alpha = d\beta$ for some β. The objective of this problem is to give a direct proof of Poincaré's lemma.

Let v be a vector field on a manifold M and α be a k-form on M. Assume that the flow φ_t of v is defined for all $t \in \mathbb{R}$.
(a) Prove that $i_v \varphi_t^* \alpha = \varphi_t^* i_v \alpha$.
(b) Prove that for all $t \in \mathbb{R}$

$$\frac{d}{dt}\varphi_t^*\alpha = L_v \varphi_t^* \alpha = \varphi_t^* L_v \alpha.$$

Let now $M = \mathbb{R}^n$ and v be given by $v(x) = -x$, $x \in \mathbb{R}^n$. (Here we identify $T_x\mathbb{R}^n$ and $\mathbb{R}^n$.) Let $k \geq 1$. Consider the linear operator $H_k\colon \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n)$ defined by

$$(H_k\alpha)(w_1, \ldots, w_{k-1}) = -\int_0^\infty (\varphi_t^* i_v \alpha)(w_1, \ldots, w_{k-1})\, dt$$

for a k-form α on $\mathbb{R}^n$.

Over, please!

(c) Prove that the improper integral in the definition of $H_k\alpha$ converges.

(d) Show that $dH_k\alpha + H_{k+1}d\alpha = \alpha$ for any k-form α on $\mathbb{R}^n$. In particular, if α is closed, $\alpha = dH_k\alpha$ and, hence, α is exact. This completes the proof of Poincaré's lemma. Hint: Use Part (b) and Cartan's formula!