MATH 209, MANIFOLDS II, WINTER 2023

Midterm
due Tuesday 2/28

1. Let X and Y be smooth vector fields on a manifold M and $\alpha \in \Omega^*(M)$. Prove that $L_X(i_Y\alpha) = i_{[X,Y]}\alpha + i_Y(L_X\alpha)$.

2. Let α be a non-vanishing 1-form on a manifold M. Prove that the following two conditions are equivalent:

(i) For any two vector fields X and Y such that $\alpha(X) = 0$ and $\alpha(Y) = 0$ identically on M we necessarily have $\alpha([X, Y]) = 0$. (In other words, for any two vector fields X and Y taking values in $\ker\alpha$ at every point of M, the bracket $[X, Y]$ also takes values in $\ker\alpha$. Remark: A distribution, i.e., a sub-bundle of TM, meeting this requirement is called integrable or involutive; cf. Chap. 19.)

(ii) $\alpha \wedge d\alpha = 0$.

3. Consider the vector fields $v = \partial_x$ and $w = x\partial_y + \partial_z$ on $\mathbb{R}^3$. Let f be a function such that $L_v f = 0$ and $L_w f = 0$ everywhere on $\mathbb{R}^3$. Prove that f is a constant function.

4. Let B be a smooth function on $\mathbb{R}^2$ with coordinates q_1 and q_2. Consider the two-form

$$\omega = B \cdot dq_1 \wedge dq_2 + dp_1 \wedge dq_1 + dp_2 \wedge dq_2$$

on $T^*\mathbb{R}^2 = \mathbb{R}^4$. Let also X_H be the Hamiltonian vector field of $H = (p_1^2 + p_2^2)/2$ on $\mathbb{R}^4$ with respect to ω. Recall that X_H is defined by the equation $i_{X_H}\omega = -dH$.

(a) Prove that ω is a symplectic form.

(b) Show that

$$X_H = p_1\partial_{q_1} + p_2\partial_{q_2} + B \cdot (p_2\partial_{p_1} - p_1\partial_{p_2}).$$

(c) Prove that the projection $q(t) = (q_1(t), q_2(t))$ of an integral curve of X_H to $\mathbb{R}^2$ satisfies the equation $\ddot{q} = -B(q)J\dot{q}$, where J is the rotation in $\pi/2$ counterclockwise. Furthermore, $p_1(t) = \dot{q}_1(t)$ and $p_2(t) = \dot{q}_2(t)$.

(d) Assume that B is constant. Show that the solutions of the equation from (c) are circles. Calculate the radii of these circles as a function of the energy $E = (p_1^2(0) + p_2^2(0))/2$ and the constant B.

Remark. By Part (c), this problem gives a Hamiltonian description of the motion of a unit charge (with unit mass) confined to a plane in the magnetic field of magnitude B, perpendicular to the plane.

5. Let ω be an n-form on an n-dimensional manifold M. Assume that $\omega_p \neq 0$ at some point $p \in M$. Show that there exist local coordinates $x_1, \ldots, x_n$ near p such that $\omega = dx_1 \wedge \cdots \wedge dx_n$.

6. Prove that for any $n \geq 1$ the manifold $S^n \times S^1$ is parallelizable, i.e., its tangent bundle is trivial.