

MATH 209, MANIFOLDS II, WINTER 2023

Homework Assignment IV: Differential Forms

Throughout this assignment we assume that M a smooth manifold of dimension n .

1. Prove that for a vector field v on M and $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$, we have $i_v(\alpha \wedge \beta) = (i_v\alpha) \wedge \beta + (-1)^k \alpha \wedge (i_v\beta)$.

Remark. This is a consequence of a similar identity in linear algebra: $i_v(\alpha \wedge \beta) = (i_v\alpha) \wedge \beta + (-1)^k \alpha \wedge (i_v\beta)$, where $v \in V$ and $\alpha \in \wedge^k V^*$ and $\beta \in \wedge^l V^*$.

2. Prove that every k -form ω on M can be written as a (locally finite) sum of products of compactly supported one-forms. More precisely, there exist compactly supported one-forms α_i^j with i ranging within some countable set and $j = 1, \dots, k$ such that the sets $F_i = \cup_j \text{supp}(\alpha_i^j)$ form a locally finite cover of M and

$$\omega = \sum_i \alpha_i^1 \wedge \dots \wedge \alpha_i^k.$$

Hint. Here is one possible approach. Using a partition of unity associated with a locally finite cover by coordinate charts reduce the question to the case where ω is supported in a coordinate chart U with coordinates (x_1, \dots, x_n) . Then, use a cut-off function equal to one on $\text{supp}(\omega)$ and vanishing outside of U to extend each dx_l to a smooth one form β_l on M equal to dx_l near $\text{supp}\omega$ and vanishing outside of U .

3. Let $F: M \rightarrow N$ be a smooth map. As we have seen, it induces the pull-back map of algebras $F^*: \Omega^*(N) \rightarrow \Omega^*(M)$. Prove that $dF^* = F^*d$.

Hint. Observe that, utilizing the properties of d and F and Problem 2, it is enough to prove this for functions and one-forms.

4. Problem 14-6 (page 375) from Chapter 14 of the textbook.

5. The goal of this problem is to show that *grad*, *curl*, and *div* are just particular cases of the de Rham differential on $M = \mathbb{R}^3$. Let us equip \mathbb{R}^3 with the standard inner product $\langle \cdot, \cdot \rangle$ and denote by \mathcal{X} the space of smooth vector fields on \mathbb{R}^3 . Define:

- $\Psi_1: \mathcal{X} \rightarrow \Omega^1(\mathbb{R}^3)$ by $\Psi_1(v) = \langle v, \cdot \rangle$ or, more explicitly, $\Psi_1(a\partial_x + b\partial_y + c\partial_z) = a dx + b dy + c dz$, where $\partial_x := \partial/\partial x$, etc.
- $\Psi_2: \mathcal{X} \rightarrow \Omega^2(\mathbb{R}^3)$ by $\Psi_2(v) = i_v(dx \wedge dy \wedge dz)$. (Work out an explicit expression for Ψ_2 .)
- $\Psi_3: C^\infty(\mathbb{R}^3) \rightarrow \Omega^3(\mathbb{R}^3)$ by $\Psi_3(f) = f dx \wedge dy \wedge dz$.

Prove that the following diagram commutes (up to signs):

$$\begin{array}{ccccccc} C^\infty(\mathbb{R}^3) & \xrightarrow{\text{grad}} & \mathcal{X} & \xrightarrow{\text{curl}} & \mathcal{X} & \xrightarrow{\text{div}} & C^\infty(\mathbb{R}^3) \\ \downarrow \text{id} & & \downarrow \Psi_1 & & \downarrow \Psi_2 & & \downarrow \Psi_3 \\ C^\infty(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3) \end{array}$$

Remark. The first and the last square of the diagram make sense (how?) and commute for \mathbb{R}^n , but the middle one does not. Note also that in \mathbb{R}^3 the identities $\text{curl} \circ \text{grad} = 0$ and $\text{div} \circ \text{curl} = 0$ together are equivalent to $d^2 = 0$.