

## MATH 209, MANIFOLDS II, WINTER 2023

### Homework Assignment III: Linear Algebra

Throughout this assignment, we use the notation and conventions used in class rather than in the textbook. In particular,  $V$  is always assumed to be a real vector space of dimension  $n$  and  $e_1, \dots, e_n$  is a basis in  $V$ . Furthermore,  $A^k(V) = \bigwedge^k V^*$  (see Problem 7) stands for the linear space of skew-symmetric multi-linear maps  $\underbrace{V \times \dots \times V}_{k \text{ times}} \rightarrow \mathbb{R}$ .

1. Prove that the elements  $\delta_I = e_{i_1}^* \wedge \dots \wedge e_{i_k}^*$  with  $I = \{i_1 < \dots < i_k\}$  introduced in class do form a basis of  $A^k(V) = \bigwedge^k V^*$ .
2. Prove that the wedge product is associative.
3. Let  $\alpha_1, \dots, \alpha_n$  be elements of  $V^*$ . Define the matrix  $a_{ij}$  by  $\alpha_i = \sum_j a_{ij} e_j^*$ . Prove that  $\alpha_1 \wedge \dots \wedge \alpha_n = \det(a_{ij}) e_1^* \wedge \dots \wedge e_n^*$ .
4. Let  $F: V \rightarrow V$  be a linear map. Prove that the map of one-dimensional vector spaces  $F^*: \bigwedge^n V^* \rightarrow \bigwedge^n V^*$  is multiplication by  $\det F$ .

Remark. Recall in this connection, that a linear map  $A: E \rightarrow E$  from a one-dimensional vector space *to itself* is always multiplication by a well-defined constant  $a$ . (This constant is defined by  $A(v) = a \cdot v$  for all  $v \in E$  or any particular  $v \neq 0$ .) On the other hand, a linear map between different one-dimensional spaces is multiplication by a constant that is not well-defined, i.e., depends on the choice of bases. This explains why the determinant is well-defined for linear maps from a space to itself, but not for maps between different vector spaces of the same dimension.

5. An element  $\omega \in A^2(V) = \bigwedge^2 V^*$  is called *non-degenerate* or a *linear symplectic form* on  $V$  if  $i_v \omega \neq 0$  for any non-zero  $v \in V$ . Define the matrix  $A = (a_{ij})$  by  $\omega = \sum_{i,j} a_{ij} e_i^* \otimes e_j^*$ . Note that  $A$  is skew-symmetric, i.e.,  $A^\top = -A$ , since  $\omega$  is skew-symmetric. (You may want to verify this...)
  - (a) Consider the map  $V \rightarrow V^*$  given by  $v \mapsto i_v \omega$ . Prove that this map is an isomorphism if and only if  $\omega$  is non-degenerate. Furthermore,  $A$  the matrix of this map in the bases  $\{e_i\}$  and  $\{e_i^*\}$ . Thus,  $\omega$  is non-degenerate if and only if  $\det A \neq 0$ .
  - (b) Prove that  $n = \dim V$  is necessarily even, say  $n = 2m$ , provided that  $V$  admits a linear symplectic form.
  - (c) Prove that  $\omega$  is non-degenerate if and only if  $\omega^m \neq 0$  in  $\bigwedge^n V^*$ .

6. Consider the map  $\Phi: V \otimes W^* \rightarrow L(W, V)$  sending  $v \otimes \alpha$ , where  $v \in V$  and  $\alpha \in W^*$ , to the linear map  $W \ni x \mapsto \alpha(x)v \in V$ . Prove that  $\Phi$  is an isomorphism.

Remark. Since the definition of this map does not involve any other structures, this map is canonical. (On a more formal level, *canonical* means an equivalence of functors...)

7. Define canonical isomorphisms between  $L^k(V)$  and  $(V^*)^{\otimes k}$ , and between  $A^k(V)$  and  $\bigwedge^k V^*$ .