MATH 209, MANIFOLDS II, WINTER 2023

Homework Assignment I: One-forms and integration

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $y = x^3$. Calculate $f_*(\partial/\partial x)$ and $f^*\,dy$. Is $f_*(\partial/\partial x)$ smooth?

2. Let f be a smooth function on a manifold M. Show that $df = f^*dy$, where y is the natural coordinate on $\mathbb{R}$. Furthermore, let $(x_1, \dots, x_n)$ be a system of local coordinates on M. Prove that in local coordinates

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

Remark. Here one has to be careful with definitions. In this problem, we define the one-form df by $df(v) = L_v f$. The one-forms $dx_1, \dots, dx_n$ are defined as the basis in $T^*_x M$ dual to the basis $\partial/\partial x_1, \dots \partial/\partial x_n$ in $T_x M$. (Note that alternatively one could define dx_i by $dx_i(v) = L_v x_i$, where x_i is now thought of as a function defined on an open subset of M. It follows from the assertion of the problem that this definition is equivalent to the one through the dual basis.)

3. A one-form α on a smooth connected manifold M is said to be *exact* if there exists a function f such that $\alpha = df$. Prove that

- α is exact $\Longleftrightarrow \int_\gamma \alpha = 0$ for any closed curve γ.

Remark. Here is a hint for the part $\Leftarrow$). The goal is to construct f such that $\alpha = df$. Fix a reference point $x_0 \in M$. For $x \in M$, set $f(x) = \int_\eta \alpha$, where η is curve connecting x_0 to x. First show that the function f is well defined, i.e., $f(x)$ is independent of the curve η connecting x_0 and x, using the assumption that $\int_\gamma \alpha = 0$. Then prove that $df = \alpha$. (For instance, one can use local coordinates to this end.)

4. Let $\alpha = x\,dy$ on the plane $\mathbb{R}^2$ with coordinates (x, y). Denote by S^1_R the circle of radius $R > 0$ centered at the origin, oriented counter clockwise. Evaluate

$$\int_{S^1_R} \alpha.$$

Conclude from the result that α is not exact. More generally, let γ be a closed simple curve in $\mathbb{R}^2$. What is the geometrical meaning of $\int_\gamma x\,dy$?

5. Problems 11-4, 11-7(c), 11-10 (Chapter 11) on pages 299–301 of the textbook.