MATH 209, MANIFOLDS II, WINTER 2019

Homework Assignment II: Vector bundles

1. Let E be a vector bundle over M. Show that for any two points $p \neq q$ in M there exists a section s of E such that $s(p) = 0$ and $s(q) \neq 0$. (In contrast with the case of smooth or continuous functions on M, we cannot replace p and q here by disjoint closed sets. Why?)

2. Let $E_{\mathbb{C}}$ be a complex line bundle and let $E_{\mathbb{R}}$ be the same line bundle regarded as a real vector bundle. (Thus $\operatorname{rk}_{\mathbb{R}} E_{\mathbb{R}} = 2$.) Prove that $E_{\mathbb{C}}$ is trivial if and only if $E_{\mathbb{R}}$ is trivial.

3. Let E be a real vector bundle over a compact manifold M. Assume that $\operatorname{rk} E > \dim M$. Prove that E has a non-vanishing section.

Remark. Hint: use the transversality theorem. Even if you cannot quite prove the assertion, try to come up with a plausible reasoning and figure out what it is exactly that you are not proving.

4. Problem 10-5 (Chapter 10) on page 269 of the textbook.

Remark. You may also want to take a look at Problems 10-6, 10-12, and 10-7. (The latter is rather tedious.) These problems will give you a better understanding of how transition functions work.

5. Problems 10-1, 10-13, and 10-16 (Chapter 10) on pages 268–271 of the textbook.

Remark. Note that a smooth vector bundle which is trivial as a smooth vector bundle is also trivial as a continuous vector bundle. Then in 10-1, it suffices to show that E is not trivial as a continuous vector bundle.

6. Prove that the tautological line bundle E over $\mathbb{R}P^n = \mathrm{Gr}_{\mathbb{R}}(1, n+1)$ is non-trivial. What is the unit sphere bundle SE in this case? Identify SE and the projection $SE \to \mathbb{R}P^n$.

Hint. First observe that the restriction of a trivial vector bundle to any subset (submanifold) is necessarily trivial. Then show that the restriction of E to $\mathrm{Gr}_{\mathbb{R}}(1, 2)$ is the tautological line bundle over $\mathrm{Gr}_{\mathbb{R}}(1, 2)$, which is non-trivial by Problem 4.

7*. Let E be the tautological complex line bundle over $\mathbb{C}P^n = \mathrm{Gr}_{\mathbb{C}}(1, n+1)$.

(a) Prove that SE is diffeomorphic to S^{2n+1}.

(b) Prove furthermore that the natural projection $SE \to \mathbb{C}P^n$ is (what does this mean?) the Hopf bundle $S^{2n+1} \to \mathbb{C}P^n$.

(c) Sketch the fibers of the Hopf bundle $S^3 \to \mathbb{C}P^1$ in $\mathbb{R}^3 = S^3 \setminus \{\text{one point}\}$.

Remark. This is an important problem which relies on some definitions that you may need to look up or ask me about, e.g., the definition of the Hopf bundle, although I have discussed some of them in class. Note the similarity with Problem 5. Also, $\mathbb{C}P^1$ is diffeomorphic to S^2. Why? Define a homeomorphism if not a diffeomorphism. If we accept the fact that S^3 is not homeomorphic to $S^2 \times S^1$ or more generally that S^{2n+1} is not homeomorphic to $\mathbb{C}P^n \times S^1$ (e.g., because $\pi_1(S^{2n+1}) = 0$ while $\pi_1(\mathbb{C}P^n \times S^1) = \mathbb{Z}$), we conclude that E is non-trivial.