MATH 209, MANIFOLDS II, WINTER 2017

Homework Assignment VI: De Rham cohomology and the Poincaré lemma

1. Problem 17-1, page 464, of the textbook.

2. Problem 17-4, page 465, of the textbook.

3. Problem 17-10, page 465, of the textbook.

4. The main objective of this problem is to prove that $H^*_c(M) = H^{*+1}_c(M\times\mathbb{R})$, where M is an arbitrary manifold – as we will soon see that this is a consequential result. Here we closely follow the argument from *Differential forms in algebraic topology* by Bott and Tu. First let us set some notation and conventions. Denote by π the natural projection $M\times\mathbb{R}\to M$ and let t be a point in $\mathbb{R}$. A k-form ω on $M\times\mathbb{R}$ can be expressed as a linear combination of forms of two types: forms of the first type are $f\cdot\pi^*\alpha$, where $\alpha\in\Omega^k(M)$ (not necessarily compactly supported) and $f\in C^\infty_c(M\times\mathbb{R})$; forms of the second type are $(\pi^*\beta)\wedge(f\,dt)$, where $\beta\in\Omega^{k-1}(M)$ (not necessarily compactly supported) and $f\in C^\infty_c(M\times\mathbb{R})$.

(a) Define the "integration over the fibers" $\pi_*\colon\Omega^*_c(M\times\mathbb{R})\to\Omega^{*-1}_c(M)$ by $\pi_*\omega=0$, when ω is of the first type, and $\pi_*(\omega)=\left(\int_{-\infty}^{\infty} f\,dt\right)\beta$ if ω is of the second type. Then π_* is a well-defined map. Prove that $d\pi_*=\pi_*d$, i.e., π_* induces a homomorphism of complexes.

(b) Fix a smooth compactly supported function $g(t)$ with $\int_{-\infty}^{\infty} g\,dt=1$. Define a linear map $\Phi\colon\Omega^{*-1}_c(M)\to\Omega^*_c(M\times\mathbb{R})$ by $\Phi(\alpha)=\alpha\wedge g\,dt$. Prove that Φ commutes with the differential d.

Our next objective is two show that π_* and Φ induce, in cohomology, maps that are inverse to each other.

(c) Define $K\colon\Omega^*_c(M\times\mathbb{R})\to\Omega^{*-1}_c(M\times\mathbb{R})$ by $K(\omega)=0$ when ω is of the first type and, when $\omega=f\,dt\wedge\pi^*\beta$ is of the second type, by

$$K(\omega)=\left(\int_{-\infty}^{t} f\,dt - G(t)\int_{-\infty}^{\infty} f\,dt\right)\cdot\beta,$$

where $G(t)=\int_{-\infty}^{t} g\,dt$. Prove that $\omega-\Phi\pi_*(\omega)=(-1)^{k-1}(dK-Kd)(\omega)$ for $\omega\in\Omega^k_c(M\times\mathbb{R})$.

(d) Prove, using (c), that π_* induces an isomorphism $H^*_c(M)\cong H^{*+1}_c(M\times\mathbb{R})$.

5. Derive from Problem 4 that

$$H^*_c(\mathbb{R}^n)=\begin{cases}0 & \text{if } *\neq n,\\ \mathbb{R} & \text{if } *=n.\end{cases}$$

This result is known as the Poincaré lemma for forms with compact support.