

MATH 209, MANIFOLDS II, WINTER 2017

Homework Assignment IV: Differential Forms

Throughout this assignment we assume that  $M$  a smooth manifold of dimension  $n$ .

1. Prove that for a vector field  $v$  on  $M$  and  $\alpha \in \Omega^k(M)$  and  $\beta \in \Omega^l(M)$ , we have  $i_v(\alpha \wedge \beta) = (i_v\alpha) \wedge \beta + (-1)^k \alpha \wedge (i_v\beta)$ .

Remark. This is a consequence of a similar identity in linear algebra:  $i_v(\alpha \wedge \beta) = (i_v\alpha) \wedge \beta + (-1)^k \alpha \wedge (i_v\beta)$ , where  $v \in V$  and  $\alpha \in \wedge^k V^*$  and  $\beta \in \wedge^l V^*$ .

2. Prove that every  $k$ -form  $\omega$  on  $M$  can be written as a (locally finite) sum of products of compactly supported one-forms. More precisely, there exist compactly supported one-forms  $\alpha_i^j$  with  $i$  ranging within some countable set and  $j = 1, \dots, k$  such that the sets  $F_i = \cup_j \text{supp}(\alpha_i^j)$  form a locally finite cover of  $M$  and

$$\omega = \sum_i \alpha_i^1 \wedge \dots \wedge \alpha_i^k.$$

Hint. Here is one possible approach. Using a partition of unity associated with a locally finite cover by coordinate charts reduce the question to the case where  $\omega$  is supported in a coordinate chart  $U$  with coordinates  $(x_1, \dots, x_n)$ . Then, use a cut-off function equal to one on  $\text{supp}(\omega)$  and vanishing outside of  $U$  to extend each  $dx_l$  to a smooth one form  $\beta_l$  on  $M$  equal to  $dx_l$  near  $\text{supp}\omega$  and vanishing outside of  $U$ .

3. Let  $F: M \rightarrow N$  be a smooth map. As we have seen, it induces the pull-back map of algebras  $F^*: \Omega^*(N) \rightarrow \Omega^*(M)$ . Prove that  $dF^* = F^*d$ .

Hint. Observe that, utilizing the properties of  $d$  and  $F$  and Problem 2, it is enough to prove this for functions and one-forms.

4. Problem 14-6 (page 375) from Chapter 14 of the textbook.

5. The goal of this problem is to show that *grad*, *curl*, and *div* are just particular cases of the de Rham differential on  $M = \mathbb{R}^3$ . Let us equip  $\mathbb{R}^3$  with the standard inner product  $\langle \cdot, \cdot \rangle$  and denote by  $\mathcal{X}$  the space of smooth vector fields on  $\mathbb{R}^3$ . Define:

- $\Psi_1: \mathcal{X} \rightarrow \Omega^1(\mathbb{R}^3)$  by  $\Psi_1(v) = \langle v, \cdot \rangle$  or, more explicitly,  $\Psi_1(a\partial_x + b\partial_y + c\partial_z) = a dx + b dy + c dz$ .
- $\Psi_2: \mathcal{X} \rightarrow \Omega^2(\mathbb{R}^3)$  by  $\Psi_2(v) = i_v(dx \wedge dy \wedge dz)$ . (Work out an explicit expression for  $\Psi_2$ .)
- $\Psi_3: C^\infty(\mathbb{R}^3) \rightarrow \Omega^3(\mathbb{R}^3)$  by  $\Psi_3(f) = f dx \wedge dy \wedge dz$ .

Prove that the following diagram commutes (up to signs):

$$\begin{array}{ccccccc} C^\infty(\mathbb{R}^3) & \xrightarrow{\text{grad}} & \mathcal{X} & \xrightarrow{\text{curl}} & \mathcal{X} & \xrightarrow{\text{div}} & C^\infty(\mathbb{R}^3) \\ \downarrow \text{id} & & \downarrow \Psi_1 & & \downarrow \Psi_2 & & \downarrow \Psi_3 \\ C^\infty(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3) \end{array}$$

Remark. The first and the last square of the diagram make sense (how?) and commute for  $\mathbb{R}^n$ , but the middle one does not. Note also that in  $\mathbb{R}^3$  the identities  $\text{curl} \circ \text{grad} = 0$  and  $\text{div} \circ \text{curl} = 0$  together are equivalent to  $d^2 = 0$ .