MATH 209, MANIFOLDS II, WINTER 2017

Final
due Monday, 03/20, by 3pm

Throughout the exam all manifolds, maps, and homotopies are assumed to be smooth.

1. Prove that the complex projective space $\mathbb{C}P^n$ is orientable.

2. Let A be the surface area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ with respect to the induced metric. Prove that $A = nV$, where V be the volume of the unit ball in $\mathbb{R}^n$. Hint: show that the Riemannian volume form on S^{n-1} (or the area form if you wish to distinguish it from the form on $\mathbb{R}^n$) is the restriction to S^{n-1} of the form

$$\sigma = \sum_{j=1}^{n} (-1)^{j-1} x_j \, dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n,$$

defined on $\mathbb{R}^n$, and then use Stokes' theorem.

3. Let α be a closed 1-form on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, set

$$f(x) = \int_\gamma \alpha,$$

where γ is the straight path connecting 0 to x and oriented from 0 to x, i.e., $\gamma(t) = tx$ for $t \in [0, 1]$. Prove that $df = \alpha$. (This is a very particular case of Poincaré's lemma.)

4. Let M be a compact connected n-dimensional manifold with non-empty boundary. Prove that $H^n(M) = 0$.

Remark. More generally, $H^n_c(M) = 0$ once $\partial M \neq \emptyset$ even when M is not compact.

5. Problem 17-7 on page 465 of the textbook: Let M_1 and M_2 be connected smooth manifolds of dimension $n \geq 3$ and let $M_1 \# M_2$ be their smooth connected sum. (See Example 9.31 in the textbook for the definition.) Prove that $H^k(M_1 \# M_2) = H^k(M_1) \oplus H^k(M_2)$ when $0 < k < n-1$ and also for $k = n-1$ provided that M_1 and M_2 are both compact and orientable. Show, that, as a consequence, $H^1(\Sigma_g) = \mathbb{R}^{2g}$, where Σ_g is the sphere with g handles.

6. (Hopf invariant) The goal of this problem is to introduce a homotopy invariant of maps $f\colon S^3 \to S^2$, the so-called Hopf invariant, and to establish some of its properties.

(a) Let $\omega \in \Omega^2(S^2)$ be an arbitrary form with $\int_{S^2} \omega = 1$. Since $H^2(S^3) = 0$, the pull back $f^*\omega$ is exact. Define the Hopf invariant of f by $H(f) = \int_{S^3} \lambda \wedge f^*\omega$, where λ is a primitive of $f^*\omega$, i.e., $d\lambda = f^*\omega$. Prove that $H(f)$ is well-defined, i.e., independent of the choice of λ when ω is fixed and, furthermore, independent of the choice of ω.

(b) Evaluate the Hopf invariant for the Hopf fibration $S^3 \to S^2$.

(c) Prove that $H(f)$ is homotopy invariant.

Remark. One can also show that $H(f) \in \mathbb{Z}$ for all f and $f_0 \sim f_1$ iff $H(f_0) = H(f_1)$.

7. Prove that a real vector bundle over S^1 is trivial if and only if it is orientable.