MATH 209, MANIFOLDS II, WINTER 2015

Final
due Tuesday 3/17

Throughout the exam all manifolds, maps, and homotopies are assumed to be smooth.

1. Prove that the complex projective space $\mathbb{C}P^n$ is orientable.

2. Let D be a closed two-dimensional disk.

(a) Show that there is no smooth map $f\colon D \to \partial D$ such that $f|_{\partial D} = id$.
(b) Let $f\colon D \to \mathbb{R}^2$ be a smooth map such that the origin $p = (0,0)$ is not in $f(\partial D)$ and the winding number $W(p, f|_{\partial D}) \neq 0$. Prove that $f^{-1}(p) \neq \emptyset$.

3. Use Problem 2(b) to prove the fundamental theorem of algebra, i.e., the fact that a complex polynominal $f(z) = z^k + a_1 z^{k-1} + \cdots$ of degree $k \geq 1$ has at least one complex root. (Of course, f has k roots counting with multiplicity. The latter result, which you don't need to prove here, follows from the existence of at least one root by the Euclid algorithm.)

Hint: Consider the family of maps $\gamma_t\colon S^1 \to \mathbb{C}$ given by $\gamma_t(\theta) = f_t\big(Re^{2\pi i\theta}\big)$ where $R > 0$ is sufficiently large and $f_t(z) = (1-t)f(z) + tz^k$, $t \in [0, 1]$. Prove that $0 \notin \gamma_t(S^1)$ for all $t \in [0, 1]$ and use this family to show that $W(0, \gamma_0) = k$. Then apply 2(b) to establish that f has a root in the disk of radius R centered at 0.

4. Let A be the surface area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$ with respect to the induced metric. Prove that $A = nV$, where V be the volume of the unit ball in $\mathbb{R}^n$. Hint: show that the Riemannian volume form on S^{n-1} (or the area form if you wish to distinguish it from the form on $\mathbb{R}^n$) is the restriction to S^{n-1} of the form

$$\sigma = \sum_{j=1}^{n} (-1)^{j-1} x_j \, dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n,$$

defined on $\mathbb{R}^n$, and then use Stokes' theorem.

5. Let α be a closed 1-form on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, set

$$f(x) = \int_\gamma \alpha,$$

where γ is the straight path connecting 0 to x and oriented from 0 to x, i.e., $\gamma(t) = tx$ for $t \in [0, 1]$. Prove that $df = \alpha$. (This is a very particular case of Poincaré's lemma.)

6. The objective of this problem is to give a direct proof of Poincaré's lemma, i.e., of the fact that every closed from on $\mathbb{R}^n$ is automatically exact. Let v be a vector field on a manifold M. Assume that the flow φ_t of v is defined for all $t \in \mathbb{R}$.

(a) Prove that $i_v \varphi_t^* \alpha = \varphi_t^* i_v \alpha$ for any differential form α.
(b) Prove that

$$\frac{d}{dt} \varphi_t^* \alpha = L_v \varphi_t^* \alpha = \varphi_t^* L_v \alpha$$

for all $t \in \mathbb{R}$ and any differential form α.

Let now $M = \mathbb{R}^n$ and v be given by $v(x) = -x$, $x \in \mathbb{R}^n$. (Here we identify $T_x\mathbb{R}^n$ and $\mathbb{R}^n$.) Let $k \geq 1$. Consider the linear operator $H_k\colon \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n)$ defined by

$$(H_k\alpha)(w_1, \ldots, w_{k-1}) = -\int_0^\infty (\varphi_t^* i_v \alpha)(w_1, \ldots, w_{k-1}) \, dt$$

for a k-form α on $\mathbb{R}^n$.

Over, please!

(c) Prove that the improper integral in the definition of $H_k\alpha$ converges.

(d) Complete the proof of Poincaré's lemma by showing that $dH_k\alpha + H_{k+1}d\alpha = \alpha$ for any k-form α on $\mathbb{R}^n$. (Thus, if α is closed, $\alpha = dH_k\alpha$ and, hence, α is exact.) Hint: Use Part (b) and Cartan's formula!