

MATH 209, MANIFOLDS II, WINTER 2011

Homework Assignment V: Orientations and integration

1. Let $F: S^n \rightarrow S^n$ be the antipodal map.
 - (a) Prove that F is orientation preserving when n is odd and orientation reversing when n is even.
 - (b) Prove that $\mathbb{R}P^n$ is orientable if and only if n is odd.
2. Let $F: \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic function. Show that F is necessarily orientation preserving at its regular points, i.e., $F^*dx \wedge dy = f dx \wedge dy$ with $f \geq 0$.
3. Let N be a hypersurface in M and let ω be a volume form on M .
 - (a) Let v be a vector field nowhere tangent to N . Prove that $i_v\omega|_N$ is a volume form on N .
 - (b) Prove that $i_v\omega|_N = i_w\omega|_N$ if $v - w$ is tangent to N .

Remark. Assume that the hypersurface N is the boundary of M . Then the construction of Part (a) gives an alternative description of the orientation induced on N . Indeed, let v point outward and let an orientation of M be determined by ω . Then the induced orientation of $N = \partial M$ is determined by $i_v\omega$ and is well defined. Note also that in both (a) and (b) it suffices to have v defined only along N .

4. Let $M \subset \mathbb{R}^3$ be the graph of a function $z = f(x, y)$ with (x, y) lying in some bounded closed domain U of \mathbb{R}^2 . Let \mathbf{v} be a vector field in \mathbb{R}^3 defined on a neighborhood of M . Recall from vector calculus that the surface integral of \mathbf{v} over M is defined as

$$\iint_M \mathbf{v} \cdot d\mathbf{S} = \iint_M \mathbf{v} \cdot \mathbf{n} dS = \iint_U (\mathbf{v} \cdot \mathbf{n})(x, y) \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2} dx dy,$$

where \mathbf{n} is the unit upward normal vector field to M . Let $\omega = dx \wedge dy \wedge dz$.

- (a) Prove that $\iint_M \mathbf{v} \cdot d\mathbf{S} = \int_M i_{\mathbf{v}}\omega$, where the orientation of M is induced by ω and \mathbf{n} as in Problem 3.
- (b) Prove that $F^*i_{\mathbf{n}}\omega = \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2} dx \wedge dy$, where $F: U \rightarrow M$ is the natural diffeomorphism $(x, y) \mapsto (x, y, f(x, y))$.

Remark. Since every hypersurface is locally a graph (in some orthogonal coordinates), this statement indicates that in general the integral of \mathbf{v} over a hypersurface in the sense of vector calculus is equal to the integral of $i_{\mathbf{v}}\omega$.

5. Problem 15-3, page 407 of the textbook.
- 6*. Prove the fundamental theorem of algebra as outlined in class.