MATH 209, MANIFOLDS II, WINTER 2011

Homework Assignment III: Linear Algebra

Throughout this assignment, we utilize the notation and conventions used in class rather than in the textbook. In particular, V is always assumed to be a real vector space of dimension n and $e_1, \dots, e_n$ is a basis in V. Furthermore, $A^k(V) = \bigwedge^k V^*$ (see Problem 7) stands for the linear space of skew-symmetric multi-linear maps $\underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}$.

1. Prove that the elements $\delta_I = e^*_{i_1} \wedge \cdots \wedge e^*_{i_k}$ with $I = \{i_1 < \ldots < i_k\}$ introduced in class do form a basis of $A^k(V) = \bigwedge^k V^*$.

2. Prove that the wedge product is associative.

3. Let $\alpha_1, \dots, \alpha_n$ be elements of V^*. Define the matrix a_{ij} by $\alpha_i = \sum_j a_{ij} e^*_j$. Prove that $\alpha_1 \wedge \cdots \wedge \alpha_n = \det(a_{ij})\, e^*_1 \wedge \cdots \wedge e^*_n$.

4. Let $F\colon V \to V$ be a linear map. Prove that the map of one-dimensional vector spaces $F^*\colon \bigwedge^n V^* \to \bigwedge^n V^*$ is multiplication by $\det F$.

Remark. Recall in this connection, that a linear map $A\colon E \to E$ from a one-dimensional vector space *to itself* is always multiplication by a well-defined constant a. (This constant is defined by $A(v) = a \cdot v$ for all $v \in E$ or any particular $v \neq 0$.) On the other hand, a linear map between different one-dimensional spaces is multiplication by a constant that is not well-defined, i.e., depends on the choice of bases. This explains why the determinant is well-defined for linear maps from the space to itself, but not for maps between different vector spaces of the same dimension.

5. An element $\omega \in A^2(V) = \bigwedge^2 V^*$ is called *non-degenerate* or a *linear symplectic form* on V if $i_v\omega \neq 0$ for any non-zero $v \in V$. Define the matrix $A = (a_{ij})$ by $\omega = \sum_{i,j} a_{ij}\, e^*_i \otimes e^*_j$. Note that A is skew-symmetric, i.e., $A^\top = -A$, since ω is skew-symmetric. (You may want to verify this...)

(a) Consider the map $V \to V^*$ given by $v \mapsto i_v\omega$. Prove that this map is an isomorphism if and only if ω is non-degenerate. Furthermore, A the matrix of this map in the bases $\{e_i\}$ and $\{e^*_i\}$. Thus, ω is non-degenerate if and only if $\det A \neq 0$.

(b) Prove that $n = \dim V$ is necessarily even, say $n = 2m$, provided that V admits a linear symplectic form.

(c) Prove that ω is non-degenerate if and only if $\omega^m \neq 0$ in $\bigwedge^n V^*$.

6. Consider the map $\Phi\colon V \otimes W^* \to L(W, V)$ sending $v \otimes \alpha$, where $v \in V$ and $\alpha \in W^*$, to the linear map $W \ni x \mapsto \alpha(x)v \in V$. Prove that Φ is an isomorphism.

Remark. Since the definition of this map does not involve any other structures, this map is canonical. (On a more formal level, *canonical* means an equivalence of functors...)

7. Define canonical isomorphisms between $L^k(V)$ and $(V^*)^{\otimes k}$, and between $A^k(V)$ and $\bigwedge^k V^*$.