

MATH 209, MANIFOLDS II, WINTER 2010

Midterm
due Tuesday 2/23

1. Let X and Y be smooth vector fields on a manifold M and $\alpha \in \Omega^*(M)$. Prove that $L_X(i_Y\alpha) = i_{[X,Y]}\alpha + i_Y(L_X\alpha)$.
2. Let φ_t be the flow generated by a smooth vector field X on M . Let $\alpha \in \Omega^*(M)$. Prove that $L_X\alpha = 0$ iff $\varphi_t^*\alpha = \alpha$.
3.
 - (a) Let α be a non-vanishing one-form on M and let ω be a two-form such that $\alpha \wedge \omega = 0$. Prove that there exists a one-form λ such that $\omega = \lambda \wedge \alpha$.
 - (b) Let \mathcal{D} be an involutive distribution of codimension-one on M which can be given as the kernel of a non-vanishing form α , i.e., $\mathcal{D}_x = \ker \alpha_x$ for all $x \in M$. (In other words, \mathcal{D} is co-orientable.) Then, as we have seen in the previous homework assignment, $\alpha \wedge d\alpha = 0$. By Part (a), there exists a one-form λ such that $d\alpha = \lambda \wedge \alpha$. Consider the three-form $\eta = \lambda \wedge d\lambda$. Prove that η is closed and that η is uniquely determined by \mathcal{D} up to an exact form. In other words, up to an exact three-form, η is independent of the choice of α and the choice of λ (none of which is unique).

Hints and remarks. In part (a), you may first show that there exists a vector field Z such that $\alpha(Z) \equiv 1$ and then set $\lambda = -i_Z\omega$. Regarding part (b), note that the distribution \mathcal{D} determines α only up to multiplication by a non-vanishing function and α determines λ up to a form $f\alpha$. By part (b), the cohomology class of η is well defined. This is the so-called Godbillion–Vey class of \mathcal{D} .

4. Let B be a smooth function on \mathbb{R}^2 with coordinates q_1 and q_2 . Consider the two-form

$$\omega = B \cdot dq_1 \wedge dq_2 + dp_1 \wedge dq_1 + dp_2 \wedge dq_2$$

on $T^*\mathbb{R}^2 = \mathbb{R}^4$. Let also X_H be the Hamiltonian vector field of $H = (p_1^2 + p_2^2)/2$ on \mathbb{R}^4 with respect to ω . Recall that X_H is defined by the equation $i_{X_H}\omega = -dH$.

- (a) Prove that ω is a symplectic form.
- (b) Show that

$$X_H = p_1\partial_{q_1} + p_2\partial_{q_2} + B \cdot (p_2\partial_{p_1} - p_1\partial_{p_2}).$$

- (c) Prove that the projection $q(t) = (q_1(t), q_2(t))$ of an integral curve of X_H to \mathbb{R}^2 satisfies the equation $\ddot{q} = -B(q)J\dot{q}$, where J is the rotation in $\pi/2$ counterclockwise. Furthermore, $p_1(t) = \dot{q}_1(t)$ and $p_2(t) = \dot{q}_2(t)$.
- (d) Assume that B is constant. Show that the solutions of the equation from (c) are circles. Calculate the radii of these circles as a function of the energy $E = (p_1^2(0) + p_2^2(0))/2$ and the constant B .

Remark. By Part (c), this problem gives a Hamiltonian description of the motion of a unit charge (with unit mass) confined to a plane in the magnetic field of magnitude B , perpendicular to the plane.

5. Show that the total space of the tangent bundle TM is an orientable manifold, regardless of whether M itself is orientable or not. Moreover, TM carries a canonical orientation (as a manifold).
6. Prove that for any $n \geq 1$ the manifold $S^n \times S^1$ is parallelizable, i.e., its tangent bundle is trivial.