MATH 209, MANIFOLDS II, WINTER 2010

Homework Assignment VII: De Rham cohomology and the Poincaré lemma

1. Problem 15-3, page 407, of the textbook.

2. Problem 15-8, page 408, of the textbook. Remark: in part (c) use the fact that $H^2(S^n) = 0$, if $n \neq 2$, which we will soon prove.

3. The main objective of this problem is to prove that $H_c^*(M) = H_c^{*+1}(M \times \mathbb{R})$, where M is an arbitrary manifold – as we will soon see that this is a consequential result. Here we closely follow the argument from *Differential forms in algebraic topology* by Bott and Tu. First let us set some notation and conventions. Denote by π the natural projection $M \times \mathbb{R} \to M$ and let t be a point in $\mathbb{R}$. A k-form ω on $M \times \mathbb{R}$ can be expressed as a linear combination of forms of two types: forms of the first type are $f \cdot \pi^*\alpha$, where $\alpha \in \Omega^k(M)$ (not necessarily compactly supported) and $f \in C_c^\infty(M \times \mathbb{R})$; forms of the second type are $(\pi^*\beta) \wedge (f\,dt)$, where $\beta \in \Omega^{k-1}(M)$ (not necessarily compactly supported) and $f \in C_c^\infty(M \times \mathbb{R})$.

(a) Define the "integration over the fibers" $\pi_*\colon \Omega_c^*(M \times \mathbb{R}) \to \Omega_c^{*-1}(M)$ by $\pi_*\omega = 0$, when ω is of the first type, and $\pi_*(\omega) = \left(\int_{-\infty}^{\infty} f\,dt\right)\beta$ if ω is of the second type. Then π_* is a well-defined map. Prove that $d\pi_* = \pi_* d$, i.e., π_* induces a homomorphism of complexes.

(b) Fix a smooth compactly supported function $g(t)$ with $\int_{-\infty}^{\infty} g\,dt = 1$. Define a linear map $\Phi\colon \Omega_c^{*-1}(M) \to \Omega_c^*(M \times \mathbb{R})$ by $\Phi(\alpha) = \alpha \wedge g\,dt$. Prove that Φ commutes with the differential d.

Our next objective is two show that π_* and Φ induce, in cohomology, maps that are inverse to each other.

(c) Define $K\colon \Omega_c^*(M \times \mathbb{R}) \to \Omega_c^{*-1}(M \times \mathbb{R})$ by $K(\omega) = 0$ when ω is of the first type and, when $\omega = f\,dt \wedge \pi^*\beta$ is of the second type, by

$$K(\omega) = \left(\int_{-\infty}^{t} f\,dt - G(t)\int_{-\infty}^{\infty} f\,dt\right) \cdot \beta,$$

where $G(t) = \int_{-\infty}^{t} g\,dt$. Prove that $\omega - \Phi\pi_*(\omega) = (-1)^{k-1}(dK - Kd)(\omega)$ for $\omega \in \Omega_c^k(M \times \mathbb{R})$.

(d) Prove, using (c), that π_* induces an isomorphism $H_c^*(M) \cong H_c^{*+1}(M \times \mathbb{R})$.

4. Derive from Problem 4 that

$$H_c^*(\mathbb{R}^n) = \begin{cases} 0 & \text{if } * \neq n, \\ \mathbb{R} & \text{if } * = n. \end{cases}$$

This result is known as the Poincaré lemma for forms with compact support.