# MATH 209, MANIFOLDS II, WINTER 2010

## Homework Assignment VI: Orientations and integration

**1.** Let $F\colon S^n \to S^n$ be the antipodal map.

(a) Prove that $F$ is orientation preserving when $n$ is odd and orientation reversing when $n$ is even.

(b) Prove that $\mathbb{R}P^n$ is orientable if and only if $n$ is odd.

**2.** Let $F\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Show that $F$ is necessarily orientation preserving at its regular points, i.e., $F^*dx \wedge dy = f\, dx \wedge dy$ with $f \geq 0$.

**3.** Let $N$ be a hypersurface in $M$ and let $\omega$ be a volume form on $M$.

(a) Let $v$ be a vector field nowhere tangent to $N$. Prove that $i_v\omega|_N$ is a volume form on $N$.

(b) Prove that $i_v\omega|_N = i_w\omega|_N$ if $v - w$ is tangent to $N$.

Remark. Assume that the hypersurface $N$ is the boundary of $M$. Then the construction of Part (a) gives an alternative description of the orientation induced on $N$. Indeed, let $v$ point outward and let an orientation of $M$ be determined by $\omega$. Then the induced orientation of $N = \partial M$ is determined by $i_v\omega$ and is well defined. Note also that in both (a) and (b) it suffices to have $v$ defined only along $N$.

**4.** Let $M \subset \mathbb{R}^3$ be the graph of a function $z = f(x, y)$ with $(x, y)$ lying in some bounded closed domain $U$ of $\mathbb{R}^2$. Let $\mathbf{v}$ be a vector field in $\mathbb{R}^3$ defined on a neighborhood of $M$. Recall from vector calculus that the surface integral of $\mathbf{v}$ over $M$ is defined as

$$\iint_M \mathbf{v} \cdot d\mathbf{S} = \iint_M \mathbf{v} \cdot \mathbf{n}\, dS = \iint_U (\mathbf{v} \cdot \mathbf{n})(x, y)\sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}\, dx\, dy,$$

where $\mathbf{n}$ is the unit upward normal vector field to $M$. Let $\omega = dx \wedge dy \wedge dz$.

(a) Prove that $\iint_M \mathbf{v} \cdot d\mathbf{S} = \int_M i_{\mathbf{v}}\omega$, where the orientation of $M$ is induced by $\omega$ and $\mathbf{n}$ as in Problem 3.

(b) Prove that $F^* i_{\mathbf{n}}\omega = \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}\, dx \wedge dy$, where $F\colon U \to M$ is the natural diffeomorphism $(x, y) \mapsto (x, y, f(x, y))$.

Remark. Since every hypersurface is locally a graph (in some orthogonal coordinates), this statement indicates that in general the integral of $\mathbf{v}$ over a hypersurface in the sense of vector calculus is equal to the integral of $i_{\mathbf{v}}\omega$.

**5.** Let $\omega \in \Omega^2(M)$ and $u\colon [0, 1] \times [0, 1] \to M$ be a smooth map. Prove that $u^*\omega = \omega\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s}\right) dt \wedge ds$ and, as a consequence,

$$\int_u \omega = \int_0^1 \int_0^1 \omega\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s}\right) dt\, ds.$$

**6.** Let $\omega = x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy$. Evaluate $\int_{S^2_R} \omega$, where $S^2_R$ is the sphere of radius $R$ centered at the origin.