MATH 209, MANIFOLDS II, WINTER 2010

Homework Assignment IV: Differential Forms

Throughout this assignment we assume that M a smooth manifold of dimension n.

1. Prove that for a vector field v on M and $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$, we have $i_v(\alpha \wedge \beta) = (i_v\alpha) \wedge \beta + (-1)^k\alpha \wedge (i_v\beta)$.

Remark. This is a consequence of a similar identity in linear algebra: $i_v(\alpha \wedge \beta) = (i_v\alpha) \wedge \beta + (-1)^k\alpha \wedge (i_v\beta)$, where $v \in V$ and $\alpha \in \bigwedge^k V^*$ and $\beta \in \bigwedge^l V^*$.

2. Prove that every k-form ω on M can be written as a (locally finite) sum of products of compactly supported one-forms. More precisely, there exist compactly supported one-forms α_i^j with i ranging within some countable set and $j = 1, \ldots, k$ such that the sets $F_i = \cup_j supp\,(\alpha_i^j)$ form a locally finite cover of M and

$$\omega = \sum_i \alpha_i^1 \wedge \cdots \wedge \alpha_i^k.$$

Hint. Here is one possible approach. Using a partition of unity associated with a locally finite cover by coordinate charts reduce the question to the case where ω is supported in a coordinate chart U with coordinates $(x_1, \ldots, x_n)$. Then, use a cut-off function equal to one on $supp\,(\omega)$ and vanishing outside of U to extend each dx_l to a smooth one form β_l on M equal to dx_l near $supp\,\omega$ and vanishing outside of U.

3. Let $F\colon M \to N$ be a smooth map. As we have seen, it induces the pull-back map of algebras $F^*\colon \Omega^*(N) \to \Omega^*(M)$. Prove that $dF^* = F^*d$.

Hint. Observe that, utilizing the properties of d and F and Problem 2, it is enough to prove this for functions and one-forms.

4. Problem 12-6 (page 320) from Chapter 12 of the textbook.

5. The goal of this problem is to show that $grad$, $curl$, and div are just particular cases of the de Rham differential on $M = \mathbb{R}^3$. Let us equip $\mathbb{R}^3$ with the standard inner product $\langle \cdot, \cdot \rangle$ and denote by $\mathcal{V}$ the space of smooth vector fields on $\mathbb{R}^3$. Define:

- $\Psi_1\colon \mathcal{V} \to \Omega^1(\mathbb{R}^3)$ by $\Psi_1(v) = \langle v, \cdot \rangle$ or, more explicitly, $\Psi_1(a\partial_x + b\partial_y + c\partial_z) = a\,dx + b\,dy + c\,dz$.
- $\Psi_2\colon \mathcal{V} \to \Omega^2(\mathbb{R}^3)$ by $\Psi_1(v) = i_v(dx \wedge dy \wedge dz)$. (Work out an explicit expression for Ψ_2.)
- $\Psi_3\colon C^\infty(\mathbb{R}^3) \to \Omega^3(\mathbb{R}^3)$ by $\Psi_3(f) = f\,dx \wedge dy \wedge dz$.

Prove that the following diagram commutes (up to signs):

$$
\begin{array}{ccccccc}
C^\infty(\mathbb{R}^3) & \xrightarrow{grad} & \mathcal{V} & \xrightarrow{curl} & \mathcal{V} & \xrightarrow{div} & C^\infty(\mathbb{R}^3) \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \Psi_1} & & \downarrow{\scriptstyle \Psi_2} & & \downarrow{\scriptstyle \Psi_3} \\
C^\infty(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3)
\end{array}
$$

Remark. The first and the last square of the diagram make sense (how?) and commute for $\mathbb{R}^n$, but the middle one does not.