# MATH 209, MANIFOLDS II, WINTER 2009

**Midterm**
**due Tuesday 2/25**

**1.** Let $X$ and $Y$ be smooth vector fields on a manifold $M$ and $\alpha \in \Omega^*(M)$. Prove that $L_X(i_Y\alpha) = i_{[X,Y]}\alpha + i_Y(L_X\alpha)$.

**2.**

(a) Let $\alpha$ be a non-vanishing one-form on $M$ and let $\omega$ be a two-form such that $\alpha \wedge \omega = 0$. Prove that there exists a one-form $\lambda$ such that $\omega = \lambda \wedge \alpha$.

(b) Let $\mathcal{D}$ be an involutive distribution of codimension-one on $M$ which can be given as the kernel of a non-vanishing form $\alpha$, i.e., $\mathcal{D}_x = \ker \alpha_x$ for all $x \in M$. (In other words, $\mathcal{D}$ is co-orientable.) Then, as we have seen in the previous homework assignment, $\alpha \wedge d\alpha = 0$. By Part (a), there exists a one-form $\lambda$ such that $d\alpha = \lambda \wedge \alpha$. Consider the three-form $\eta = \lambda \wedge d\lambda$. Prove that $\eta$ is closed and that $\eta$ is uniquely determined by $\mathcal{D}$ up to an exact form. In other words, up to an exact three-form, $\eta$ is independent of the choice of $\alpha$ and the choice of $\lambda$ (none of which is unique).

Hints and remarks. In part (a), you may first show that there exists a vector field $Z$ such that $\alpha(Z) \equiv 1$ and then set $\lambda = -i_Z\omega$. Regarding part (b), note that the distribution $\mathcal{D}$ determines $\alpha$ only up to multiplication by a non-vanishing function and $\alpha$ determines $\lambda$ up to a form $f\alpha$. By part (b), the cohomology class of $\eta$ is well defined. This is the so-called Godbillion–Vey class of $\mathcal{D}$.

**3.** Let $B$ be a smooth function on $\mathbb{R}^2$ with coordinates $q_1$ and $q_2$. Consider the two-form

$$\omega = B \cdot dq_1 \wedge dq_2 + dp_1 \wedge dq_1 + dp_2 \wedge dq_2$$

on $T^*\mathbb{R}^2 = \mathbb{R}^4$. Let also $X_H$ be the Hamiltonian vector field of $H = (p_1^2 + p_2^2)/2$ on $\mathbb{R}^4$ with respect to $\omega$. Recall that $X_H$ is defined by the equation $i_{X_H}\omega = -dH$.

(a) Prove that $\omega$ is a symplectic form.

(b) Show that

$$X_H = p_1\partial_{q_1} + p_2\partial_{q_2} + B \cdot (p_2\partial_{p_1} - p_1\partial_{p_2}).$$

(c) Prove that the projection $q(t) = (q_1(t), q_2(t))$ of an integral curve of $X_H$ to $\mathbb{R}^2$ satisfies the equation $\ddot{q} = -B(q)J\dot{q}$, where $J$ is the rotation in $\pi/2$ counterclockwise. Furthermore, $p_1(t) = \dot{q}_1(t)$ and $p_2(t) = \dot{q}_2(t)$.

(d) Assume that $B$ is constant. Show that the solutions of the equation from (c) are circles. Calculate the radii of these circles as a function of the energy $E = (p_1^2(0) + p_2^2(0))/2$ and the constant $B$.

Remark. By Part (c), this problem gives a Hamiltonian description of the motion of a unit charge (with unit mass) confined to a plane in the magnetic field of magnitude $B$, perpendicular to the plane.

**4.** Show that the total space of the tangent bundle $TM$ is an orientable manifold, regardless of whether $M$ itself is orientable or not. Moreover, $TM$ carries a canonical orientation (as a manifold).

**5*.** Let $M$ be a closed orientable surface of genus $g$. Prove that there exist $2g$ closed one-forms $\alpha_1, \ldots, \alpha_{2g}$ on $M$ such that any linear combination $a_1\alpha_1 + \cdots + a_{2g}\alpha_{2g}$ with constant coefficients is exact if and only if $a_1 = \ldots = a_{2g} = 0$. (Remark: in this problem you need to use the classification of closed surfaces. Note also that this is not a very easy problem although solving it does not require any advanced understanding of manifolds beyond what has been discussed in class.)