

MATH 209, MANIFOLDS II, WINTER 2009

Homework Assignment V: More on Differential Forms, due Wednesday 2/18

Throughout this assignment we assume that M a smooth manifold of dimension n .

1. Prove, using the original definition, that for a vector field v on M and $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$, we have $L_v(\alpha \wedge \beta) = (L_v\alpha) \wedge \beta + \alpha \wedge (L_v\beta)$.

2. Let $\omega = dx_1 \wedge \cdots \wedge dx_n$ on \mathbb{R}^n . Prove that for any vector field $v = (v_1, \dots, v_n)$ on \mathbb{R}^n , we have $L_v\omega = (\operatorname{div} v)\omega$, where $\operatorname{div} v := (\partial v_1/\partial x_1) + \cdots + (\partial v_n/\partial x_n)$.

Remark. By definition, a volume form on M is an n -form ω such that $\omega_x \neq 0$ for any $x \in M$, i.e., ω is a non-vanishing n -form. Define the divergence $\operatorname{div}_\omega v$ of v with respect to ω by $L_v\omega = (\operatorname{div}_\omega v)\omega$. Then $\operatorname{div}_\omega v$ measures to what extent the flow φ_t of v is volume expanding or contracting. In particular, $\operatorname{div}_\omega v = 0$ is equivalent to $\varphi_t^*\omega = \omega$, i.e., the flow is volume preserving.

3. Recall that a distribution \mathcal{D} on M is a smooth sub-bundle of TM , i.e., a family of subspaces $\mathcal{D}_x \subset T_xM$ depending smoothly on x . A distribution is said to be involutive or integrable if for any two vector fields v and w tangent to \mathcal{D} (i.e., such that $v(x)$ and $w(x)$ are in \mathcal{D}_x for all $x \in M$), the Lie bracket $[v, w]$ is again tangent to \mathcal{D} . (See the textbook for more details.)

- (a) Is the distribution \mathcal{D} spanned by $v = \partial_x$ and $w = x\partial_z - \partial_y$ on \mathbb{R}^3 integrable? Conclude from your solution that every function which is constant along the distribution \mathcal{D} (i.e., such that $L_v f = L_w f = 0$ everywhere) must be constant. Sketch this distribution.
- (b) Let $f: M \rightarrow \mathbb{R}$ be a function without critical points on M , i.e., such that $df_x \neq 0$ for any $x \in M$. Show that the distribution $\mathcal{D}_x := \ker df_x$ is integrable. (What are the integral submanifolds for this distribution?)

4. Let α be a non-vanishing one-form on M . Consider the distribution $\mathcal{D}_x = \ker \alpha_x$. (For instance, the distribution from Problem 3(a) can be given as $\ker(dz + x dy)$.) Prove that \mathcal{D} is involutive if and only if $\alpha \wedge d\alpha = 0$.

Hint. First prove that $\alpha \wedge d\alpha = 0$ is equivalent to that $d\alpha$ vanishes on \mathcal{D} , i.e., $d\alpha(v, w) = 0$ for any vectors v and w tangent to \mathcal{D} . Then use the expression $d\alpha(v, w) = L_v\alpha(w) - L_w\alpha(v) - \alpha([v, w])$. The result of Problem 4 is a form of the Frobenius theorem for distributions of codimension one. What examples of involutive distributions of codimension one do you know? Try to construct such a distribution (or rather a foliation) on S^3 .

5. Problem 12-11 (page 321) and Problem 12-13 (page 322) from Chapter 12 of the textbook.

6*. Let φ^t and ψ^t be, respectively, the flows of vector fields v and w on M . Fix $p \in M$ and set $\gamma(t) = \psi^{-t}\varphi^{-t}\psi^t\varphi^t(p)$. Prove that $\gamma'(0) = 0$. Then show that $\gamma''(0)$ is well defined and equal to $2[v, w](p)$.

Remark. This explicitly shows that $[v, w]$ measures to what extent the flows of v and w do not commute. The difficult part is the identity $\gamma''(0) = 2[v, w](p)$. You may want to first look into this identity for linear vector fields on \mathbb{R}^n , before dealing with the general case. (See also Spivak's *A Comprehensive Introduction to Differential Geometry*, vol. I, p. 159–162.)