MATH 209, MANIFOLDS II, WINTER 2009

Final

Throughout the exam all manifolds, maps, and homotopies are assumed to be smooth.

1. Let D be a closed two-dimensional disk.

(a) Show that there is no smooth map $f\colon D \to \partial D$ such that $f\mid_{\partial D}= id$.

(b) Let $f\colon D \to \mathbb{R}^2$ be a smooth map such that the origin $p = (0, 0)$ is not in $f(\partial D)$ and the winding number $W(p, f\mid_{\partial D}) \neq 0$. Prove that $f^{-1}(p) \neq \emptyset$.

2. Show that S^{2n} with $n > 1$ does not admit a symplectic structure.

3. Problem 15-7 on page 408 of the textbook. Prove, as a consequence, that $H^1(\Sigma_g) = \mathbb{R}^{2g}$, where Σ_g is the sphere with g handles.

4. Let U and V be open subsets of a manifold M and let α be a differential form on $U \cap V$. Show that there exist differential forms β on U and γ on V such that $\alpha = \beta\mid_{U\cap V} - \gamma\mid_{U\cap V}$. (This problem completes the construction of the Mayer–Vietoris exact sequence.)

5. The objective of this problem is to give a direct proof of Poincaré's lemma. Let v be a vector field on a manifold M. Assume that the flow φ_t of v is defined for all $t \in \mathbb{R}$.

(a) Prove that $i_v\varphi_t^*\alpha = \varphi_t^* i_v\alpha$ for any differential form α.

(b) Prove that

$$\frac{d}{dt}\varphi_t^*\alpha = L_v\varphi_t^*\alpha = \varphi_t^* L_v\alpha$$

for all $t \in \mathbb{R}$ and any differential form α.

Let now $M = \mathbb{R}^n$ and v be given by $v(x) = -x$, $x \in \mathbb{R}^n$. (Here we identify $T_x\mathbb{R}^n$ and $\mathbb{R}^n$.) Let $k \geq 1$. Consider the linear operator $\Phi_k\colon \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n)$ defined by

$$(\Phi_k\alpha)(w_1, \ldots, w_{k-1}) = -\int_0^\infty (\varphi_t^* i_v\alpha)(w_1, \ldots, w_{k-1})\, dt$$

for a k-form α on $\mathbb{R}^n$.

(c) Prove that the improper integral in the definition of $\Phi_k\alpha$ converges.

(d) Complete the proof of Poincaré's lemma by showing that $d\Phi_k\alpha + \Phi_{k+1}d\alpha = \alpha$ for any k-form α on $\mathbb{R}^n$. (Thus, if α is closed, $\alpha = d\Phi_k\alpha$ and, hence, α is exact.) Hint: Use Part (b) and Cartan's formula!

6*. (Hopf invariant) The goal of this problem is to introduce a homotopy invariant of maps $f\colon S^3 \to S^2$, the so-called Hopf invariant, and to establish some of its properties.

(a) Let $\omega \in \Omega^2(S^2)$ be an arbitrary form with $\int_{S^2}\omega = 1$. Since $H^2(S^3) = 0$, the pull back $f^*\omega$ is exact. Define the Hopf invariant of f by $H(f) = \int_{S^3}\lambda \wedge f^*\omega$, where λ is a primitive of $f^*\omega$, i.e., $d\lambda = f^*\omega$. Prove that $H(f)$ is well-defined, i.e., independent of the choice of λ when ω is fixed and, furthermore, independent of the choice of ω.

(b) Evaluate the Hopf invariant for the Hopf fibration $S^3 \to S^2$.

(c) Prove that $H(f)$ is homotopy invariant.

(d) Consider maps $h\colon S^3 \to S^3$ and $g\colon S^2 \to S^2$. Prove that $H(g \circ f) = \deg(g)^2 H(f)$ and $H(f \circ h) = \deg(h)H(f)$.

Remarks. One can also show that $H(f) \in \mathbb{Z}$ for all f. It follows from Parts (b) and (d) that $H(f)$ can assume arbitrary integer values. Furthermore, (b) and (c) imply that the Hopf fibration map is not homotopic to identity. The converse of (c) also holds: two maps f_1 and f_0 from S^3 to S^2 are homotopic if and only if $H(f_0) = H(f_1)$.