MATH 208, Fall 2024

Manifolds I, Midterm

1. Let M be a smooth manifold. Prove that for any continuous function $f\colon M \to \mathbb{R}$ and any $\epsilon > 0$, there exists a smooth function h such that $\sup_{x\in M} |f(x) - h(x)| < \epsilon$. (In other words, every continuous function can be C^0-approximated by smooth functions.)

Hint: You may consider, for instance, using the Weierstrass approximating theorem asserting that every continuous function on a closed cube (or any compact set) in $\mathbb{R}^n$ can be approximated by polynomials.

2. Let M be a closed (i.e., compact without boundary) manifold of dimension n. Prove that there is no immersion $M \to \mathbb{R}^n$.

3. Consider the subset of $\mathbb{R}^4$ given by $x_1^2 + x_1^3 - x_2^2 + x_3 x_4 = c$ where $c \in \mathbb{R}$. Prove that this is a smooth submanifold of $\mathbb{R}^4$ when $c \neq 0$ and $c \neq 4/27$. Is it compact?

4. Let v and w be smooth vector fields on M and let f be a smooth function. Prove that

$$[v, fw] = (L_v f)w + f[v, w].$$

5. Denote by M_n the vector space of real $n \times n$ matrices.

(a) Prove that for any $X \in \mathrm{M}_n$ we have

$$\frac{d}{dt}\det(I + tX)\bigg|_{t=0} = \operatorname{tr} X.$$

(b) Show that 1 is a regular value of the function $\det\colon \mathrm{M}_n \to \mathbb{R}$. (Hint: reduce the problem to checking that $I \in \mathrm{M}_n$ is a regular point and then use (a).) As a consequence, prove that $\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{M}_n \mid \det A = 1\}$ is a smooth hypersurface in M_n.

Over please!

6. Consider the following vector fields on $\mathbb{R}^3$:

$$v = \frac{\partial}{\partial x} \quad \text{and} \quad w = x\frac{\partial}{\partial z} + \frac{\partial}{\partial y} \ .$$

(a) Find $[v, w]$.

(b) Assume that f is a smooth function on $\mathbb{R}^3$ such that

$$L_v f = L_w f = 0$$

at every point. Prove that $f = \mathit{const}$. Hint: first show that $L_u f = 0$ for any vector u. Then use (and prove) that

$$f(\gamma(1)) - f(\gamma(0)) = \int_0^1 L_{\dot\gamma(t)} f \, dt$$

for any smooth curve $\gamma \colon [0, 1] \to \mathbb{R}^3$.