MATH 208, MANIFOLDS I, FALL 2024

Homework Assignment 6: Sard's lemma, Whitney emmbedding theorem and all that

1. Let L and M be smooth submanifolds of $\mathbb{R}^n$. For $x \in \mathbb{R}^k$ denote by $x+L$ the submanifold L shifted by x, i.e., $x + L = \{x + y \mid y \in L\}$. Prove that $(x + L) \cap M = \emptyset$ for almost all $x \in \mathbb{R}^k$ whenever $\dim L + \dim M < k$.

2. Let M be a smooth n-dimensional submanifold of $\mathbb{R}^k$. Denote by $\rho_x \colon M \to S^{k-1}$, $x \notin M$, the radial projection of M to the sphere centered at x. Assume that $2n < k$. Show that ρ_x is an immersion for almost all $x \in \mathbb{R}^n$. Hint: consider the map $\Phi \colon TM \to \mathbb{R}^k$ sending (p, v) to $p + v$. The center x should not be in the image of this map.

3. Let $\pi \colon M \to M$ be a submersion and $L \subset M$ a smooth submanifold. Show that for almost all $q \in N$ the intersection $L_q := L \cap \pi^{-1}(q)$ is a smooth submanifold of $\pi^{-1}(q)$ and hence of M. What's $\dim L_q$?

4. Assume that M is compact. Let $F \colon M \to \mathbb{R}^k$ be a smooth embedding and let $G \colon M \to \mathbb{R}^k$ be C^1-close to F. Show that G is an embedding. Note: we have shown in class that G is an immersion when F is an immersion. So you only need to prove that G is one-to-one.

5. Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a homogeneous function of degree $m \geq 1$, i.e., $f(\lambda x) = \lambda^m f(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n$. Consider the radial vector field $v(x) = x$ on $\mathbb{R}^n$. Prove that $L_v f(x)$ exists for all $x \in \mathbb{R}^n$ and $L_v f(x) = m f(x)$. Assuming next that f is smooth, show that x is a regular point of f whenever $f(x) \neq 0$ and all $a \neq 0$ are regular values. What can you say about the points on the zero locus of f?