MATH 208, MANIFOLDS I, FALL 2024

Homework Assignment 5: Immersions, submersions and local normal forms

Throughout the assignment M stands for a smooth manifold.

1. Let f be a smooth function on a neighborhood of $0 \in \mathbb{R}^n$ such that $df_0 \neq 0$. Show that there exists a coordinate system $y_1, \ldots, y_n$ near 0 such that $f(y_1, \ldots, y_n) = f(0) + y_1$. In other words, there exists a diffeomorphism φ defined on a small neighborhood of 0 and fixing 0, such that $f \circ \varphi(x_1, \ldots, x_n) = f(0) + x_1$, where $x_1, \ldots, x_n)$ is the original coordinates on $\mathbb{R}^n$. (This does not directly follow from the rank theorem or the local normal form for submersions. Why? Problem 2 is a generalization of this result when $n = 1$.)

2. Let $f(x)$ be the germ of a smooth function on $\mathbb{R}$ at 0 such that $f(0) = f'(0) = \cdots = f^{(k-1)}(0)$ but $f^{(k)}(0) \neq 0$. Show that there exists a coordinate change $x = x(y)$ such that $f\big(x(y)\big) = \pm y^k$. Equivalently, $f \circ \varphi(y) = \pm y^k$ for some diffeomorphism φ near 0. (Hint: consider using Hadamard's lemma.)

3. In a similar vein, let v be a smooth vector field on a neighborhood of $0 \in \mathbb{R}^n$ such that $v(0) \neq 0$. Show that there exists a coordinate system $y_1, \ldots, y_n$ near 0 such that $v(y) = \partial/\partial y_1$ near 0. (This fact is yet another example of a local normal form. Strictly speaking, we do not yet know quite enough to do this problem: you need the material from Chapter 9.)

4. Show that there exists a C^∞-smooth map $\gamma \colon \mathbb{R} \to \mathbb{R}^2$ such that $\gamma(\mathbb{R})$ is the graph of the function $y = |x|$.

5. Prove the Inverse Function Theorem as a consequence of the Implicit Function Theorem and the other way around.

6. Assume that $f \colon \mathbb{R} \to \mathbb{R}$ is differentiable at 0 and $f'(0) \neq 0$ and $f(0) = 0$. Let $g \colon \mathbb{R} \to \mathbb{R}$ be such that $f \circ g = id$ near 0, i.e., $f\big(g(x)\big) = x$. Prove that g is differentiable at 0 and $g'(0) = 1/f'(0)$.

7. Let $\gamma \colon \mathbb{R} \to \mathbb{T}^2 = S^1 \times S^1$ be given by $\gamma(t) = \big(e^{2\pi i t}, e^{2\pi i a t}\big)$, where $a \notin \mathbb{Q}$. Show that $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$, and that γ is an immersion.

8. Problems 4-5, 4-6, 4-13, 5-1, 5-6, 5-9 and 5-10 of the textbook.