MATH 208, MANIFOLDS I, FALL 2024

Homework Assignment 4: Vector fields

Throughout the assignment M stands for a smooth manifold.

1. Let A be an associative algebra. Define the bracket $[\cdot\,,\cdot]$ on A by setting $[a,b] := ab - ba$. Show that $(A, [\cdot\,,\cdot])$ is a Lie algebra. (Essentially, you only need to verify the Jacobi identity.) In particular, the algebra $\mathfrak{gl}(n)$ of n-by-n matrices, real or complex, is a Lie algebra.

2. Show that the following subspaces of $\mathfrak{gl}(n)$ are Lie subalgebras: $\mathfrak{sl}(n)$ formed by matrices with zero trace, $\mathfrak{so}(n)$ of skew-symmetric matrices, and $\mathfrak{u}(n)$ of anti-self-adjoint matrices. (Note: these subspaces are not associative subalgebras of $\mathfrak{gl}(n)$.)

3. Prove the Jacobi identity for the Lie bracket of vector fields.

4. Prove that the directional derivative gives rise to a canonical isomorphism between the Lie algebra of C^∞-smooth vector fields on M and the Lie algebra of derivations of $A = C^\infty(M)$. (A derivation of A is a linear map $D\colon A \to A$ such that the product rule holds: $D(fg) = (Df)g + f(Dg)$.)

5. Calculate the Lie bracket $[v,w]$ for the following vector fields on a finite-dimensional vector space V:

- $v(x) = Ax$ and $w(x) = Bx$ are linear vector fields on V. (Here A and B are linear maps $V \to V$ and use the identification $TV = V \times V$.)
- $v(x) = Ax$ is a linear vector field and $w(x) = w_0$ is a constant vector field.

From the first point you will see that linear vector fields on V form a Lie subalgebra of the Lie algebra of all vector fields.

6. Problems 8-2, 8-16, 8-19, 8-20 on pp. 199–202 of the textbook.