MATH 208, MANIFOLDS I, FALL 2024

Homework Assignment 3: Tangent spaces

Throughout the assignment M stands for a smooth manifold.

1. Let $f\colon M \to \mathbb{R}$ be a smooth function, $p \in M$ and $v \in T_pM$. Prove that $(Df)_p(v) = L_v f$, where and $(Df)_p = f_*\colon T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$ is the push-forward map and we have identified $T_{f(p)}\mathbb{R}$ with $\mathbb{R}$.

2. Let $\gamma\colon (-\epsilon, \epsilon) \to M$ be a smooth map and $p = \gamma(0)$. Denote by t the coordinate on $\mathbb{R}$. Show that $\gamma_*(\partial/\partial t) = [\gamma]$ in T_pM. (By definition, this is $\gamma'(0)$.)

3. Show that for every $r = 0, 1, \dots, \infty$ the algebra of germs of functions $C^r(\mathbb{R}^n, 0)$ at 0, or any point, has exactly one maximal ideal. This is the ideal of functions vanishing at zero and when $r = \infty$ it is generated by $x_1, \dots, x_n$. However, unless $r = \infty$, the ideal generated by the coordinate functions $x_1, \dots, x_n$ is not maximal.

4. Show that the ideal generated by $x_1, \dots, x_n$ in the algebra of formal power series $A = \mathbb{R}[[x_1, \dots, x_n]]$ is maximal and that this is the only maximal ideal in A. This is the analogue of Problem 3 for formal power series.

5. Let M be a $C^{r\geq 2}$-manifold and I be the ideal of C^r-functions vanishing at $p \in M$. Show that there is a canonical isomorphism of vector spaces between T_pM and the dual $(I/I^2)^*$. (One can replace here the algebra of functions by the algebra of germs of functions at p.)

6. Let X be a compact topological space, e.g., a compact smooth or topological manifold, and $A = C^0(X)$. (Thus even when X is a smooth manifold we are forgetting the smooth structure at this point.) Fix $p \in X$ and let $\delta_p\colon A \to \mathbb{R}$ be the evaluation functional at p, i.e., $\delta_p(f) = f(p)$. Let D be the derivation of A ta p. In other words, $D\colon A \to \mathbb{R}$ is a linear map over $\mathbb{R}$ and the product rule holds: $D(fg) = \delta_p(f)D(g) + D(f)\delta_p(g)$. Show that D is identically zero. This provides an explanation why one needs a smooth structure to define the tangent space and why the structure of a topological manifold is not enough.

7. Problems 3.1-3.4, p. 75 of the textbook.