MATH 208, MANIFOLDS I, FALL 2024

Homework Assignment I: Topological manifolds

1. Prove that a connected topological manifold is path connected.

2. Topological invariance of dimension asserts that whenever open subsets $W \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are homeomorphic, we necessarily have $n = m$. (In particular $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic only when $n = m$. Prove this for $n = 1$. For higher dimensions this is a difficult result.) Use topological invariance of dimension to show that the dimension of a connected topological manifold is well defined.

3. Prove that the letter "Y" is not a topological manifold.

4. Prove that $\mathbb{RP}^n$ is a topological manifold. Remark: there are many ways to do this and we have talked about some of them in class. A simple way is to use the construction $\mathbb{RP}^n = S^n / \sim$.

5. Let X be a topological manifold. Assume that a finite group G acts on X freely (all stabilizers are trivial) by homeomorphisms. Show that the quotient X/G is a topological manifold. (This is a generalization of Problem 4. Why?)

6. Prove that there exists a continuous onto map $[0, 1] \to [0, 1] \times [0, 1]$. (Here one can replace the square by a cube of any dimension. Such maps are often referred to as space-filling curves.)

7. Prove that a closed disk in $\mathbb{R}^2$ and a solid closed square are homeomorphic. (Again, this is true in all dimensions.)

8. Consider the algebra of functions $A = C^0([0, 1])$. Denote by $I_x \subset A$ the set of all functions vanishing at x.

(a) Prove that I_x is a maximal ideal in A.

(b) Prove that every maximal ideal in A is of the form I_x for some $x \in [0, 1]$ and that the map $x \mapsto I_x$ is a bijection between $[0, 1]$ and the set of maximal ideals in A.

Remark: The same is true for every sufficiently nice compact topological space. (You need partition of unity...) Compactness is essential: Part (b) fails for $A = C^0(\mathbb{R})$. (Why?) However, we can replace $C^0([0, 1])$ by the algebra of differentiable functions $C^r([0, 1])$, $r = 1, 2 \ldots, \infty$.