MATH 208, Fall 2012

Manifolds I, Midterm

1. Let M be a smooth manifold. Prove that for any continuous function $f\colon M \to \mathbb{R}$ and any $\epsilon > 0$, there exists a smooth function h such that $\sup_{x\in M} |f(x) - h(x)| < \epsilon$. (In other words, every continuous function can be C^0-approximated by smooth functions.)

Hint: You may consider, for instance, using the Weierstrass approximating theorem asserting that every continuous function on a closed cube (or any compact set) in $\mathbb{R}^n$ can be approximated by polynomials.

2. Let M be a closed (i.e., compact without boundary) manifold of dimension n. Prove that there is no immersion $M \to \mathbb{R}^n$.

3. For which $c \in \mathbb{R}$, the subset given by $x_1^2 + x_1^3 - x_2^2 + x_3x_4 = c$ is a smooth submanifold of $\mathbb{R}^4$?

4. Denote by M_n the vector space of real $n \times n$ matrices.

(a) Prove that for any $X \in \mathrm{M}_n$ we have

$$\frac{d}{dt}\det(I + tX)\bigg|_{t=0} = \operatorname{tr} X.$$

(b) Show that 1 is a regular value of the function $\det\colon \mathrm{M}_n \to \mathbb{R}$. (Hint: reduce the problem to checking that $I \in \mathrm{M}_n$ is a regular point and then use (a).) As a consequence, prove that $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{M}_n \mid \det A = 1\}$ is a smooth hypersurface in M_n.

5. Consider the following vector fields on $\mathbb{R}^3$:

$$v = \frac{\partial}{\partial x} \quad \text{and} \quad w = x\frac{\partial}{\partial z} + \frac{\partial}{\partial y} \quad .$$

(a) Find $[v, w]$.

(b) Assume that f is a smooth function on $\mathbb{R}^3$ such that

$$L_v f = L_w f = 0$$

at every point. Prove that $f = \mathit{const}$. Hint: first show that $L_u f = 0$ for any vector u. Then use (and prove) that

$$f(\gamma(1)) - f(\gamma(0)) = \int_0^1 L_{\dot\gamma(t)} f \, dt$$

for any smooth curve $\gamma\colon [0,1] \to \mathbb{R}^3$.

6. Let $p \in M$ be a regular point of a smooth function $f\colon M \to \mathbb{R}$. Prove that there is a coordinate system $(x_1, \dots, x_n)$ near p such that $f(x) = f(p) + x_1$.

7. Let $p \in M$ be a critical point of a smooth function $f\colon M \to \mathbb{R}$. The Hessian $d_p^2 f\colon T_pM \times T_pM \to R$ of f at p is defined as follows. Let v and w be two vectors in T_pM. Extend these vectors to vector fields $\tilde{v}$ and $\tilde{w}$ on a neighborhood of p (i.e., $\tilde{v}$ and $\tilde{w}$ are vector fields near p with $\tilde{v}(p) = v$ and $\tilde{w}(p) = w$) and set

$$d_p^2 f(v, w) = L_{\tilde{v}} L_{\tilde{w}} f.$$

- **(a)** Prove that the extensions $\tilde{v}$ and $\tilde{w}$ exist. Prove that $d_p^2 f(v, w)$ is well defined, i.e., independent of the choice of extensions $\tilde{v}$ and $\tilde{w}$. (Why is the condition that p is a critical point essential?) Show that the bilinear form $d_p^2 f$ is symmetric.
- **(b)** Find an expression for $d_p^2 f$ in local coordinates.
- **(c)** Prove that $d_p^2 f(v) \geq 0$ for all $v \in T_pM$, when p is a local minimum of f. Conversely (almost), prove that p is a local minimum when $d_p^2 f$ is positive definite, i.e., $d_p^2 f(v) > 0$ for all non-zero $v \in T_pM$.