MATH 208, Fall 2012

Manifolds I

Final

1. Consider two vector fields $v(x) = v_0 + A(x)$ and $w(x) = w_0 + B(x)$ on $\mathbb{R}^n$, where v_0 and w_0 are constant vectors, and A and B are linear maps $\mathbb{R}^n \to \mathbb{R}^n$.

a. Find the flow φ^t of v.

b. Find the braket $[v, w]$.

2. Let $\Sigma = \Sigma_g \setminus D^2$, where Σ_g is the sphere with g handles and D^2 is a disk. Show that Σ admits an immersion into into $\mathbb{R}^2$. (It is sufficient to sketch a figure or a series of figures as a solution.)

3. Let $M^m \subset \mathbb{R}^k$ and $N^n \subset \mathbb{R}^k$ be two submanifolds of dimensions m and, respectively, n such that $m + n < k$. (For instance, M and N are two curves in $\mathbb{R}^3$.) Prover that $(x + M) \cap N = \emptyset$ for almost all, in the sense of measure theory, $x \in \mathbb{R}^k$. Here $x + M = \{x + y \mid y \in M\}$.

4. Two smooth submanifolds M_0 and M_1 of N are said to be transverse if $T_pN = T_pM_0 + T_pM_1$ for all $p \in M_0 \cap M_1$. Prove that $M_0 \cap M_1$ is a smooth submanifold of N of dimension $\dim M_0 + \dim M_1 - \dim N$, whenever M_0 and M_1 are transverse.

5. Denote by M_n the vector space of real $n \times n$ matrices and let P_n be the vector space of real symmetric $n \times n$ matrices. Consider the map $F\colon \mathrm{M}_n \to P_n$ given by $F(A) = AA^T$.

a. Show that $DF_I(X) = X + X^T$ and that $I \in \mathrm{M}_n$ is a regular point of F. (Would I still be a regular point if we replaced the target space P_n by M_n?)

b. Show that $I \in P_n$ is a regular value of F. Thus the orthogonal group $\mathrm{O}(n) = \{A \in \mathrm{M}_n \mid AA^T = I\}$ is a smooth submanifold of M_n. Find $\dim \mathrm{O}(n)$.

c. Show that $T_I\,\mathrm{O}(n)$ is the space of all skew-symmetric matrices, i.e., $T_I\,\mathrm{O}(n) = \{X \in \mathrm{M}_n \mid X + X^T = 0\}$. (This space is usually denoted by $so(n)$.)

6. Let v be a vector field on M. Denote by φ^t the (local) flow of v.

- **a.** Let $F\colon M \to N$ be a diffeomorphism. Show that $F\varphi^t F^{-1}$ is the (local) flow of F_*v. (Hint: prove that $F(\varphi^t(F^{-1}(q)))$ is the integral curve of F_*v through $q \in N$.)
- **b.** Let $F\colon M \to M$ be a diffeomorphism. Prove that $F_*v = v$ if and only if $F\varphi^t = \varphi^t F$ for all t.

7. Let M be a connected manifold without boundary. Show that for any two points p and q in M, there exists a diffeomorphism $F\colon M \to M$ such that $F(p) = q$. In other words, the group of diffeomorphisms of M acts transitively on M. (Hint: First reduce the problem to the case where p and q are contained in a relatively compact coordinate neighborhood U diffeomorphic to an open ball. Then show that there exists a vector field v on M, vanishing outside U and hence complete, such that $\varphi^1(p) = q$, where φ^t is the flow of v.)