MATH 208, Fall 2011

Manifolds I, Midterm

1. Let M be a smooth manifold. Prove that for any continuous function $f\colon M \to \mathbb{R}$ and any $\epsilon > 0$, there exists a smooth function h such that $\sup_{x\in M} |f(x) - h(x)| < \epsilon$. (In other words, every continuous function can be C^0-approximated by smooth functions.)

Hint: You may consider, for instance, using the Weierstrass approximating theorem asserting that every continuous function on a closed cube (or any compact set) in $\mathbb{R}^n$ can be approximated by polynomials.

2. Let M be a closed (i.e., compact without boundary) manifold of dimension n. Prove that there is no immersion $M \to \mathbb{R}^n$.

3. Let v and w be smooth vector fields on M and let f be a smooth function. Prove that

$$[v, fw] = (L_v f)w + f[v, w].$$

4. Denote by M_n the vector space of real $n \times n$ matrices.

(a) Prove that for any $X \in \mathrm{M}_n$ we have

$$\frac{d}{dt}\det(I + tX)\bigg|_{t=0} = \operatorname{tr} X.$$

(b) Show that 1 is a regular value of the function $\det\colon \mathrm{M}_n \to \mathbb{R}$. (Hint: reduce the problem to checking that $I \in \mathrm{M}_n$ is a regular point and then use (a).) As a consequence, we conclude that $\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{M}_n \mid \det A = 1\}$ is a smooth hypersurface in M_n.

5. Consider the following vector fields on $\mathbb{R}^3$:

$$v = \frac{\partial}{\partial x} \quad \text{and} \quad w = x\frac{\partial}{\partial z} + \frac{\partial}{\partial y} \quad .$$

(a) Find $[v, w]$.

(b) Assume that f is a smooth function on $\mathbb{R}^3$ such that

$$L_v f = L_w f = 0$$

at every point. Prove that $f = const$. Hint: first show that $L_u f = 0$ for any vector u. Then use (and prove) that

$$f(\gamma(1)) - f(\gamma(0)) = \int_0^1 L_{\dot{\gamma}(t)} f \, dt$$

for any smooth curve $\gamma\colon [0,1] \to \mathbb{R}^3$.

6. Let $p \in M$ be a critical point of a smooth function $f\colon M \to \mathbb{R}$. The Hessian $d_p^2 f\colon T_pM \times T_pM \to R$ of f at p is defined as follows. Let v and w be two vectors in T_pM. Extend these vectors to vector fields $\tilde{v}$ and $\tilde{w}$ on a neighborhood of p (i.e., $\tilde{v}$ and $\tilde{w}$ are vector fields near p with $\tilde{v}(p) = v$ and $\tilde{w}(p) = w$) and set

$$d_p^2 f(v, w) = L_{\tilde{v}} L_{\tilde{w}} f.$$

(a) Prove that the extensions $\tilde{v}$ and $\tilde{w}$ exist. Prove that $d_p^2 f(v, w)$ is well defined, i.e., independent of the choice of extensions $\tilde{v}$ and $\tilde{w}$. (Why is the condition that p is a critical point essential?) Show that the bilinear form $d_p^2 f$ is symmetric.

(b) Find an expression for $d_p^2 f$ in local coordinates.

(c) Prove that $d_p^2 f(v) \geq 0$ for all $v \in T_pM$, when p is a local minimum of f. Conversely (almost), prove that p is a local minimum when $d_p^2 f$ is positive definite, i.e., $d_p^2 f(v) > 0$ for all non-zero $v \in T_pM$.

7. Let $M^m \subset \mathbb{R}^k$ and $N^n \subset \mathbb{R}^k$ be two submanifolds of dimensions m and, respectively, n such that $m + n < k$. (For instance, M and N are two curves in $\mathbb{R}^3$.) Prover that $(x + M) \cap N = \emptyset$ for almost all, in the sense of measure theory, $x \in \mathbb{R}^k$. Here $x + M = \{x + y \,|\, y \in M\}$.