Mathematics 19B; Winter 2002; V. Ginzburg
Practice Final, Solutions

1. For each of the ten questions below, state whether the assertion is *true* or *false*:

   (a) To evaluate $\int \sqrt{a^2 - x^2}\, dx$ one should use the trigonometric substitution $x = a \sin\theta$ with $-\pi/2 \le \theta \le \pi/2$. Answer: **T.**

   (b) $\int \sinh x\, dx = -\cosh x + C$. Answer: **F.** $\int \sinh x\, dx = \cosh x + C$.

   (c) If $a_n \to 0$, then the series $\sum_{n=1}^{\infty} a_n$ converges. Answer: **F.** Example: $\sum_{n=1}^{\infty} \frac{1}{n}$.

   (d) $\int_0^1 \frac{dx}{x^2}$ converges. Answer: **F.**

   (e) To find the integral $\int x^3 e^x\, dx$ one should apply the method of integration by parts. Answer: **T.**

   (f) Assume that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ exists and $L \le 1$. Then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. Answer: **F.** To guarantee absolute convergence one should have $L < 1$.

   (g) Assume that b_n is a decreasing sequence, $b_n > 0$ for all n, and $\lim_{n\to\infty} b_n = 0$. Then the series $\sum_{n=1}^{\infty} (-1)^n b_n$ converges. Answer: **T.**

   (h) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ for all $x \neq 0$. Answer: **F.** Only for $|x| < 1$.

   (i) $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x. Answer: **T.**

   (j) Let a_n be a bounded monotonic sequence. Then a_n converges. Answer: **T.**

2. Evaluate the following indefinite integrals:

   (a) $\int \frac{x}{x^2-5x+6}\, dx$. **Solution:** Solving $x^2 - 5x + 6 = 0$ we find that

$$x^2 - 5x + 6 = (x-3)(x-2).$$

   Now we look for A and B such that

$$\frac{x}{x^2 - 5x + 6} = \frac{A}{x-3} + \frac{B}{x-2}.$$

   This is equivalent to $A(x-2) + B(x-3) = x$ or

$$\begin{aligned} A + B &= 1, \\ 2A + 3C &= 0. \end{aligned}$$

Solving for A and B, we obtain $A = 3$ and $B = -2$. Hence,

$$\frac{x}{x^2 - 5x + 6} = \frac{3}{x-3} - \frac{2}{x-2}$$

and

$$\begin{aligned} \int \frac{x}{x^2 - 5x + 6}\,dx &= \int \left(\frac{3}{x-3} - \frac{2}{x-2}\right) dx \\ &= 3\ln|x-3| - 2\ln|x-2| + C. \end{aligned}$$

(b) $\int x\,e^{2x+1}\,dx$. **Solution:** Let us use the method of integration by parts. Set $u(x) = x$ and $v'(x) = e^{2x+1}$. Then $u'(x) = 1$ and $v(x) = \frac{1}{2}e^{2x+1}$. Thus,

$$\begin{aligned} \int x\,e^{2x+1}\,dx &= \frac{x}{2}e^{2x+1} - \int \frac{1}{2}e^{2x+1}\,dx \\ &= \frac{x}{2}e^{2x+1} - \frac{1}{4}e^{2x+1} + C. \end{aligned}$$

(c) $\int \tan^3 x\,dx$. **Solution:** Recall that $\tan^2 x = \sec^2 x - 1$. Thus

$$\begin{aligned} \int \tan^3 x\,dx &= \int \tan x \tan^2 x\,dx \\ &= \int \tan x(\sec^2 x - 1)\,dx \\ &= \int \tan x \sec^2 x\,dx - \int \tan x\,dx. \end{aligned}$$

To evaluate the first integral we use the substitution $u = \tan x$ (so that $du = u'\,dx = \sec^2 x\,dx$). Thus

$$\int \tan x \sec^2 x\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{\tan^2 x}{2} + C.$$

The second integral is evaluated as follows:

$$\begin{aligned} \int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx \quad \text{substitute } u = \cos x \\ &= -\int \frac{du}{u} \\ &= -\ln|u| + C \\ &= \ln|\sec x| + C. \end{aligned}$$

Combining these two integrals we obtain:

$$\int \tan^3 x\,dx = \frac{\tan^2 x}{2} - \ln|\sec x| + C.$$

Alternatively, we can write the integral as

$$\int \frac{\sin^3 x}{\cos^3 x}\,dx = \int \frac{\sin^2 x}{\cos^3 x}\,\sin x\,dx$$

and then use the substitution $u = \cos x$. This would lead to a different expression for the integral.

3. Evaluate the following definite integrals:

   (a) $\int_0^3 \frac{x}{\sqrt{x^2+16}}\,dx$. **Solution:** Let $u = x^2 + 16$. Then $du = 2x\,dx$ and

$$\int_0^3 \frac{x}{\sqrt{x^2+16}}\,dx = \frac{1}{2}\int_{16}^{25} \frac{du}{\sqrt{u}} = \sqrt{u}\,\Big|_{16}^{25} = 5 - 4 = 1.$$

   Alternatively we could use the trigonometric substitution $x = 4\tan\theta$.

   (b) $\int_0^{\frac{\pi}{2}} \cos^6 x\,\sin^3 x\,dx$. **Solution:** We rewrite the integral as

$$\int_0^{\frac{\pi}{2}} \cos^6 x\,\sin^3 x\,dx = \int_0^{\frac{\pi}{2}} \cos^6 x\,(1-\cos^2 x)\sin x\,dx.$$

   Let $u = \cos x$ so that $du = -\sin x\,dx$. Then

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^6 x\,(1-\cos^2 x)\sin x\,dx &= -\int_1^0 u^6(1-u^2)\,du \\ &= \int_0^1 u^6(1-u^2)\,du \\ &= \left(\frac{u^7}{7} - \frac{u^9}{9}\right)\Bigg|_0^1 \\ &= \frac{1}{7} - \frac{1}{9} = \frac{2}{63}.\end{aligned}$$

   (c) $\int_0^{\pi^2} \sin\sqrt{x}\,dx$. **Solution:** Let $u = \sqrt{x}$ so that $x = u^2$ and $dx = 2u\,du$. Then

$$\int_0^{\pi^2} \sin\sqrt{x}\,dx = 2\int_0^{\pi} u\sin u\,du.$$

   Integrating by parts we obtain,

$$\begin{aligned}\int_0^{\pi} u\sin u\,du &= -u\cos u\,|_0^{\pi} + \int_0^{\pi} \cos u\,du \\ &= \pi + \sin u\,|_0^{\pi} = \pi.\end{aligned}$$

   Hence,

$$\int_0^{\pi^2} \sin\sqrt{x}\,dx = 2\pi.$$

4. Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{4-x}$, $x = 0$, $y = 0$ about the x-axis.

   Solution: The volume of the region bounded by $y = f(x)$ with $a \le x \le b$ is given by
$$V = \pi \int_a^b f(x)^2\,dx.$$
   For the region under consideration $a = 1$ and b is the solution to $f(x) = 0$, i.e., $b = 4$. Thus
$$V = \pi \int_0^4 \left(\sqrt{4-x}\right)^2\,dx = \pi \int_0^4 (4-x)\,dx = \pi \left(4x - \frac{x^2}{2}\right)\Big|_0^4 = 8\pi.$$

5. Evaluate the following improper integral: $\int_1^\infty \frac{\ln x}{x^2}\,dx$.

   Solution: By definition,
$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{r\to\infty} \int_1^r \frac{\ln x}{x^2}\,dx.$$
   Setting $u = \ln x$ and $v' = 1/x^2$ (so that $v = -1/x$) we integrate by parts
$$\begin{aligned} \int_1^r \frac{\ln x}{x^2}\,dx &= -\frac{\ln x}{x}\Big|_1^r + \int_1^r \frac{dx}{x^2} \\ &= -\frac{\ln r}{r} + 1 - \frac{1}{r} \end{aligned}$$
   By L'Hostpital's rule $\frac{\ln r}{r} \to 0$ as $r \to \infty$. Thus
$$\int_1^\infty \frac{\ln x}{x^2}\,dx = 1.$$

6. Test the following series for convergence, absolute convergence, or divergence:

   (a) $\sum_{n=2}^\infty \frac{(-1)^n}{n(\ln n)^3}$. **Solution:** By the alternating series test this series *converges.* To test for absolute convergence we need to check if the series of absolute values $\sum_{n=2}^\infty \frac{1}{n(\ln n)^3}$ converges. Let us apply the integral test: the series in question converges if and only if the improper integral
$$\int_2^\infty \frac{dx}{x(\ln x)^3}$$
   converges. Set $u = \ln x$. Then
$$\int_2^\infty \frac{dx}{x(\ln x)^3} = \int_{\ln 2}^\infty \frac{du}{u^3}.$$

The later integral converges and hence the series converges. (Note also that alternatively we can conclude that the series is converging from its absolute convergence.)

(b) $\sum_{n=0}^{\infty}(-1)^n \sin n$. **Solution:** This series diverges since $(-1)^n \sin n$ does not go to zero (in fact does not have a limit at all) as $n \to \infty$.

(c) $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$. **Solution:** Let us apply the root test:

$$\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e},$$

for $\left(1+\frac{1}{n}\right)^n \to e$ as $n \to \infty$. Since $1/e < 1$, the series converges (absolutely).

7. Find the Maclaurin series of the function $y = \frac{1}{\sqrt{1-x}}$ and determine its radius of convergence.

Solution: Let us find the derivatives of $f(x) = \frac{1}{\sqrt{1-x}} = (1-x)^{-\frac{1}{2}}$:

$$\begin{aligned}
f(x) &= (1-x)^{-\frac{1}{2}} \\
f'(x) &= \frac{1}{2}(1-x)^{-\frac{3}{2}} \\
f''(x) &= \frac{1}{2}\cdot\frac{3}{2}(1-x)^{-\frac{5}{2}} \\
f^{(3)}(x) &= \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}(1-x)^{-\frac{7}{2}} \\
&\cdots \\
f^{(n)}(x) &= \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{2n-1}{2}(1-x)^{-\frac{2n+1}{2}} \quad \text{for } n \geq 1 \\
&= \frac{1\cdot 2\cdots(2n-1)}{2^n}(1-x)^{-\frac{2n+1}{2}} \quad \text{for } n \geq 1.
\end{aligned}$$

Thus we have $f(0) = 1$ and

$$f^{(n)}(0) = \frac{1\cdot 2\cdots(2n-1)}{2^n}$$

for $n \geq 1$. Recall that the the Maclaurin series has the form

$$\sum_{0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots.$$

For the function in question, we obtain

$$1 + \sum_{n=1}^{\infty}\frac{1\cdot 2\cdots(2n-1)}{2^n n!}x^n.$$

To determine the radius of convergence we use the ratio test. After all cancelations, we have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{1}{2} \cdot \frac{2n+1}{n+1}|x| \to |x| \quad \text{as } n \to \infty.$$

Hence, the series converges (absolutely) if $|x| < 1$ and diverges if $|x| > 1$. The radius of convergence is $R = 1$.