Mathematics 19B; Winter 2002; V. Ginzburg
Practice Midterm II, Solutions

1. For each of the ten questions below, state whether the assertion is *true* or *false*:

   (a) The volume of the solid obtained by rotating the region bounded by $y = f(x)$, $x = a$, and $x = b$ about the x-axis is equal to $\pi \int_a^b f(x)^2\,dx$. Answer: **T.**

   (b) The work done in moving the object from a to b is equal to $\int_a^b f(x)\,dx$, where $f(x)$ is the force. Answer: **T.**

   (c) The integration by parts formula reads

   $$\int f'(x)g(x)\,dx = f(x)g(x) + \int f(x)g'(x)\,dx.$$

   Answer: **F:** $\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$.

   (d) $\int \tan x\,dx = \ln(\sec x) + C$. Answer: **F:** $\int \tan x\,dx = \ln|\sec x| + C$.

   (e) The average value of a function $y = f(x)$ on the interval $[a,b]$ is $\int_a^b f(x)\,dx$. Answer: **F:** The average is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

   (f) To find the integral $\int e^x \sin x\,dx$ one should apply the method of integration by parts. Answer: **T.**

   (g) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$. Answer: **T.**

   (h) To evaluate $\int \sqrt{x^2-a^2}\,dx$ one should use the trigonometric substitution $x = a\sin\theta$. Answer: **F:** The correct substitution is $x = a\sec\theta$ (where $0 \le \theta < \pi/2$ or $\pi \le \theta 3\pi/2$).

   (i) $\int \ln x\,dx = x\ln x + C$. Answer: **F:** $\int \ln x\,dx = x\ln x - x + C$

   (j) $\int \sec x\,dx = \ln|\sec x + \tan x| + C$. Answer: **T.**

2. Evaluate the following indefinite integrals:

   (a) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$. **Solution:** Let us use the substitution $u = \sqrt{x}$. Then $du = \frac{dx}{2\sqrt{x}}$ and

   $$\begin{aligned} \int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx &= 2\int \sin u\,du \\ &= -2\cos u + C \\ &= -2\cos\sqrt{x} + C. \end{aligned}$$

(b) $\int x^2 \ln x\,dx$. **Solution:** Let us use the method of integration by parts. Set $f(x) = \ln x$ and $g'(x) = x^2$. Then $f'(x) = 1/x$ and $g(x) = x^3/3$. Thus,

$$\begin{aligned}
\int x^2 \ln x\,dx &= \frac{x^3 \ln x}{3} - \int \frac{x^3}{3} (\ln x)'\,dx \\
&= \frac{x^3 \ln x}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x}\,dx \\
&= \frac{x^3 \ln x}{3} - \int \frac{x^2}{3}\,dx \\
&= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C.
\end{aligned}$$

(c) $\int \sin^2 x \cos^2 x\,dx$. **Solution:** We use half-angle formulas:

$$\begin{aligned}
\sin^2 x &= \frac{1}{2}(1 - \cos 2x), \\
\cos^2 x &= \frac{1}{2}(1 + \cos 2x).
\end{aligned}$$

Then

$$\begin{aligned}
\sin^2 x \cos^2 x &= \frac{1}{4}(1 - \cos 2x)(1 + \cos 2x) \\
&= \frac{1}{4}(1 - \cos^2 2x) \\
&= \frac{1}{4}\left(1 - \frac{1}{2}(1 + \cos 4x)\right) \\
&= \frac{1}{8}(1 - \cos 4x).
\end{aligned}$$

Therefore,

$$\int \sin^4 x\,dx = \int \frac{1}{8}(1 - \cos 4x)\,dx = \frac{x}{8} - \frac{1}{32}\sin 4x + C.$$

(d) $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$. **Solution:** We use the trigonometric substitution $x = 2\sin\theta$, where $-\pi/2 \leq \theta \leq \pi/2$. Then $dx = 2\cos\theta\,d\theta$ and $\sqrt{4-x^2} = 2\cos\theta$. Hence,

$$\begin{aligned}
\int \frac{x^2}{\sqrt{4-x^2}}\,dx &= \int \frac{4\sin^2\theta}{2\cos\theta} 2\cos\theta\,d\theta \\
&= 4\int \sin^2\theta\,d\theta \\
&= 2\int (1 - \cos 2\theta)\,d\theta \\
&= 2\theta - \sin 2\theta + C \\
&= 2\theta - 2\sin\theta\cos\theta + C.
\end{aligned}$$

Furthermore, since $x = 2\sin\theta$, we have $\cos\theta = \frac{\sqrt{4-x^2}}{2}$. Hence,

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx = 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + C.$$

(e) $\int \frac{3x^2-7x-2}{x^3-x}\,dx$. **Solution:** First observe that $x^3 - x = x(x-1)(x+1)$. We need to find A, B, and C such that the integrand decomposes as

$$\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \frac{3x^2-7x-2}{x^3-x}.$$

This is equivalent to

$$A(x-1)(x+1) + Bx(x+1) + Cx(x-1) = 3x^2 - 7x - 2.$$

Expanding the left hand side, we obtain

$$(A+B+C)x^2 + (B-C)x - A = 3x^2 - 7x - 2.$$

Thus, we have the following system of equations:

$$\begin{aligned} A+B+C &= 3 \\ B-C &= -7 \\ -A &= -2 \end{aligned}$$

which yields $A = 2$, $B = -3$, and $C = 4$. Hence,

$$\frac{3x^2-7x-2}{x^3-x} = \frac{2}{x} - \frac{3}{x-1} + \frac{4}{x+1}$$

and

$$\int \frac{3x^2-7x-2}{x^3-x}\,dx = 2\ln|x| - 3\ln|x-1| + 4\ln|x+1| + C.$$

(f) $\int \frac{dx}{x(x^2+1)}\,dx$. **Solution:** We look for a decomposition of the integrand of the form

$$\frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{1}{x(x^2+1)}.$$

which is equivalent to

$$A(x^2+1) + (Bx+C)x = 1.$$

As above, expanding the left hand side and solving for A, B, and C, we obtain: $A = 1$, $B = -1$, and $C = 0$. Hence,

$$\frac{1}{x(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1}.$$

Integrating, we conclude that

$$\int \frac{dx}{x(x^2+1)}\,dx = \ln|x| - \frac{1}{2}\ln(x^2+1) + C = \ln\frac{|x|}{\sqrt{x^2+1}} + C.$$

3. Evaluate the following definite integrals:

   (a) $\int_0^{\frac{\pi}{2}} e^{\sin x}\cos x\,dx$. **Solution:** Let $u = \sin x$. Then

   $$\int_0^{\frac{\pi}{2}} e^{\sin x}\cos x\,dx = \int_0^1 e^u\,du = e - 1.$$

   (b) $\int_0^1 x\tan^{-1}x\,dx$. **Solution:** Let $u = \tan^{-1}x$ and $dv = x\,dx$. Then $du = \frac{dx}{1+x^2}$ and $v = x^2/2$. Integrating by parts we have

   $$\int x\tan^{-1}x\,dx = \frac{x^2\,\tan^{-1}x}{2} - \frac{1}{2}\int\frac{x^2}{1+x^2}\,dx.$$

   Furthermore, $\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$, and hence

   $$\int\frac{x^2}{1+x^2}\,dx = x - \tan^{-1}x + C.$$

   As a result,

   $$\int_0^1 x\tan^{-1}x\,dx = \left(\frac{x^2\,\tan^{-1}x}{2} - \frac{x}{2} + \frac{1}{2}\tan^{-1}x\right)\Bigg|_0^1 = \frac{\pi}{4} - \frac{1}{2}.$$

   (c) $\int_{-1}^1 x^3\sqrt{x^2+4}\,dx$. **Solution:** We could use the trigonometric substitution $x = 2\tan\theta$, where $-\pi/2 < \theta < \pi/2$. However, there is a simpler solution: the integrand is an odd function and the interval is symmetric. Hence, $\int_{-1}^1 x^3\sqrt{x^2+4}\,dx = 0$.

   (d) $\int_{-1}^4 \frac{x-19}{x^2-3x-10}\,dx$. **Solution:** First observe that $x^2-3x-10 = (x+2)(x-5)$. We need to find A and B such that the integrand decomposes as

   $$\frac{A}{x+2} + \frac{B}{x-5} = \frac{x-19}{x^2-3x-10}.$$

   This is equivalent to

   $$A(x-5) + B(x+2) = x - 19.$$

   Expanding the left hand side, we obtain

   $$(A+B)x + (-5A+2B) = x - 19.$$

Thus, we have the system of equations:

$$\begin{aligned} A + B &= 1 \\ -5A + 2B &= -19 \end{aligned}$$

which yields $A = 3$ and $B = -2$. Hence,

$$\frac{x - 19}{x^2 - 3x - 10} = \frac{3}{x + 2} - \frac{2}{x - 5}$$

and

$$\int_{-1}^{4} \frac{x - 19}{x^2 - 3x - 10}\, dx = (3 \ln |x + 2| - 2 \ln |x - 5|)|_{-1}^{4} = 5 \ln 6.$$