Mathematics 19B; Winter 2002; V. Ginzburg
Practice Midterm I, Solutions

1. For each of the ten questions below, state whether the assertion is *true* or *false*:

   (a) $\int \cos x\, dx = \sin x + C$. Answer: **T.**

   (b) $\int_2^7 \cos 3x\, dx = \frac{1}{3}(\sin 3x + 13)|_2^7$. Answer: **T.**

   (c) $\int_a^b f(g(x))g'(x)\, dx = \int_a^b f(u)\, du$. Answer: **F.** The correct formula:

$$\int_a^b f(g(x))g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

   (d) The area of the region bounded by the curves $y = x^3 - 14x^2 - 8x + 4$, $y = 6x^2 - 5$, $x = 3$, and $x = 7$ is $-\frac{143}{12}$. Answer: **F.** The area is never negative.

   (e) If $\int_a^b f(x)\, dx$ is negative, then $f(x_*) < 0$ at some point $a \le x_* \le b$. Answer: **T.**

   (f) $\frac{d}{dx}\int_0^x f(t)\, dt = f(x)$, provided that $y = f(x)$ is a continuous function. Answer: **T.**

   (g) $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n \sin\left(\frac{\pi j}{n}\right) = \int_0^\pi \sin x\, dx$. Answer: **T.**

   (h) $\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n \left(1 + \frac{j}{n}\right)^2 = \int_0^1 x^2\, dx$. Answer: **F.** The limit is $\int_1^2 x^2\, dx$.

   (i) Let $f(t)$ be the rate of growth of a population. Then $\int_a^b f(t)\, dt = F(b) - F(a)$, where $F(t)$ is the population as a function of time. Answer: **T.**

   (j) $\int_0^x \sin t\, dt = -\cos x$. Answer: **F.** The correct formula: $\int_0^x \sin t\, dt = -(\cos x - \cos 0) = 1 - \cos x$.

2. Evaluate the following indefinite integrals:

   (a) $\int \cos\left(x + \frac{\pi}{2}\right) dx$.
   Solution: Let $u = g(x) = x + \frac{\pi}{2}$. Then $g'(x) = 1$. By the substitution rule,

$$\int \cos\left(x + \frac{\pi}{2}\right) dx = \int \cos u\, du = \sin u + C = \sin\left(x + \frac{\pi}{2}\right) + C.$$

   (b) $\int e^{3x} dx$.
   Solution: Let $u = g(x) = 3x$. Then $g'(x) = 3$. By the substitution rule,

$$\int e^{3x}\, dx = \frac{1}{3}\int e^{3x} 3\, dx = \frac{1}{3}\int e^u\, du = \frac{1}{3}e^u + C = \frac{1}{3}e^{3x} + C.$$

(c) $\int \frac{x}{x^2+2} dx$.
Solution: Let $u = g(x) = x^2 + 2$. Then $g'(x) = 2x$. By the substitution rule,

$$\int \frac{x}{x^2+2} dx = \frac{1}{2} \int \frac{2x}{x^2+2} dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln \left(x^2+2\right) + C.$$

3. Evaluate the following definite integrals:

(a) $\int_0^1 \left(y^3-1\right)^2 y^2 dy$.
Solution: Let $u = g(y) = y^3 - 1$. Then $g'(x) = 3y^2$. By the substitution rule (for the definite integral),

$$\int_0^1 \left(y^3-1\right)^2 y^2 dy = \frac{1}{3} \int_{-1}^0 u^2 du = \frac{1}{9} u^3 \Big|_{-1}^0 = \frac{0-(-1)}{9} = \frac{1}{9}.$$

(b) $\int_1^0 \frac{1}{3-x} dx$.
Solution: Let $u = g(y) = 3 - x$. Then $g'(x) = -1$. By the substitution rule (for the definite integral),

$$\int_1^0 \frac{1}{3-x} dx = -\int_2^3 \frac{1}{u} du = -(\ln 3 - \ln 2) = \ln(2/3).$$

(c) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(x + \frac{\pi}{2}\right) dx$.
Solution: Let $u = g(y) = x + \pi/2$. Then $g'(x) = 1$. By the substitution rule (for the definite integral),

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(x + \frac{\pi}{2}\right) dx = \int_0^\pi \cos u \, du = \sin \pi - \sin 0 = 0.$$

4. Evaluate the following derivatives:

(a) $\frac{d}{dx} \int_x^1 t^3 \ln t \, dt$.
Solution: Let $F(t) = \int t^3 \ln t \, dt$. Then $F'(t) = t^3 \ln t$ and $\int_x^1 t^3 \ln t \, dt = F(1) - F(x)$. Thus,

$$\frac{d}{dx} \int_x^1 t^3 \ln t \, dt = -F'(x) = -x^3 \ln x.$$

(b) $\frac{d}{dx} \int_1^{x^3} t^2 \cos t \, dt$.

Solution: Let $F(t) = \int t^2 \cos t\, dt$. Then $F'(t) = t^2 \cos t$ and

$$\int_1^{x^3} t^2 \cos t\, dt = F(x^3) - F(1).$$

Thus

$$\frac{d}{dx} \int_1^{x^3} t^2 \cos t\, dt = \frac{d}{dx} \big(F(x^3) - F(1)\big) = \frac{d}{dx} F(x^3).$$

By the chain rule,

$$\frac{d}{dx} F(x^3) = F'(x^3)(x^3)' = (x^3)^2 \cos x^3 \cdot 3x^2 = 3x^8 \cos x^3.$$

5. Sketch the region bounded by the curves $y = \sin x$ and $y = -x$, $x = 0$, $x = \pi$, and compute the area of the region.

   Solution: The region is as follows:

   The area of the region is

$$\begin{aligned} \int_0^{\pi} \sin x\, dx - \int_0^{\pi} (-x)\, dx &= -\cos x\big|_0^{\pi} + \frac{x^2}{2}\bigg|_0^{\pi} \\ &= \cos 0 - \cos \pi + \frac{\pi^2}{2} \\ &= 2 + \frac{\pi^2}{2}. \end{aligned}$$