Mathematics 19A; Fall 2001; V. Ginzburg
Practice Midterm I; Solutions

1. For each of the ten questions below, state whether the assertion is *true* or *false*.

   (a) The function $f(x) = \sqrt[3]{x}$ is continuous at 0.
   Answer: **T**.

   (b) $\lim_{x\to 0} x\sin(\frac{1}{x}) = 0$.
   Answer: **T**.

   (c) Let $f(x)$ and $g(x)$ be continuous at $x = a$. Then the function $f(x)g(x)$ is continuous at $x = a$.
   Answer: **T**.

   (d) $\lim_{x\to -\infty}(x^7 + 2x) = \infty$.
   Answer: **F**; $\lim_{x\to -\infty}(x^7 + 2x) = -\infty$.

   (e) Let $f(x) = x^3 - x$. Then the equation $f(x) = 1$ has a solution on the interval $(0, 2)$.
   Answer: **T**; use the intermediate value theorem.

   (f) $\lim_{t\to -\infty} e^t = 0$.
   Answer: **T**.

   (g) Assume that $f(x) \geq 0$ and $\lim_{x\to a} f(x) = L$. Then $\lim_{x\to a}\sqrt{f(x)} = \sqrt{L}$.
   Answer: **T**.

   (h) $\lim_{t\to 0}\frac{1}{t} = \infty$.
   Answer: **F**; the limit does not exist: $\lim_{t\to 0-}\frac{1}{t} = -\infty$ and $\lim_{t\to 0+}\frac{1}{t} = \infty$.

   (i) The function
   $$f(x) = \frac{x+1}{|x|+1}$$
   has only one horizontal asymptote $y = 1$.
   Answer: **F**; this function has two horizontal asymptotes: $y = 1$ and $y = -1$.

   (j) The slope of the tangent to the graph of a function $f(x)$ at the point $P(a, f(a))$ is
   $$m = \lim_{x\to a}\frac{f(x) - f(a)}{x - a}.$$
   Answer: **T**.

   (k) The function
   $$f(x) = \begin{cases} x^2 + 2 & \text{for } x \leq 1, \\ 7x - 4 & \text{for } x > 1. \end{cases}$$
   is continuous at $x = 1$.
   Answer: **T**; $\lim_{x\to 1} f(x) = 3 = f(1)$.

2. Let
$$f(x) = \begin{cases} -x & \text{for } x \leq 0, \\ 2x - 1 & \text{for } 0 < x < 1, \\ 1 & \text{for } x \geq 1. \end{cases}$$

(a) Sketch the graph of $f(x)$ accurately using the solid/open circles where appropriate.

(b) From the graph of $f(x)$ find the following limits. (If the limit does not exist, indicate so.)

i. $\lim_{x\to 0-} f(x)$. Answer: $\lim_{x\to 0-} f(x) = 0$.

ii. $\lim_{x\to 0+} f(x)$. Answer: $\lim_{x\to 0+} f(x) = -1$.

iii. $\lim_{x\to 0} f(x)$. Answer: $\lim_{x\to 0} f(x)$ does not exist.

iv. $\lim_{x\to 1-} f(x)$. Answer: $\lim_{x\to 1-} f(x) = 1$.

v. $\lim_{x\to 1+} f(x)$. Answer: $\lim_{x\to 1+} f(x) = 1$.

vi. $\lim_{x\to 1} f(x)$. Answer: $\lim_{x\to 1} f(x) = 1$.

3. Evaluate the following limits

(a) $\lim_{x\to\sqrt{3}} \frac{x^4-9}{x^2-3}$.

Solution: For $x \neq \sqrt{3}$, we have

$$\frac{x^4-9}{x^2-3} = \frac{(x^2+3)(x^2-3)}{x^2-3} = x^2+3.$$

Thus

$$\lim_{x\to\sqrt{3}} \frac{x^4-9}{x^2-3} = \lim_{x\to\sqrt{3}} (x^2+3) = \lim_{x\to\sqrt{3}} (x^2) + 3 = (\sqrt{3})^2 + 3 = 6.$$

(b) $\lim_{t\to 0} \frac{\sqrt{2-t}-\sqrt{2}}{t}$.

Solution: For $t \neq 0$, we have

$$\begin{aligned}
\frac{\sqrt{2-t}-\sqrt{2}}{t} &= \frac{\left(\sqrt{2-t}-\sqrt{2}\right)\left(\sqrt{2-t}+\sqrt{2}\right)}{t\left(\sqrt{2-t}+\sqrt{2}\right)} \\
&= \frac{\left(\sqrt{2-t}\right)^2-\left(\sqrt{2}\right)^2}{t\left(\sqrt{2-t}+\sqrt{2}\right)} \\
&= \frac{2-t-2}{t\left(\sqrt{2-t}+\sqrt{2}\right)} \\
&= \frac{-1}{\sqrt{2-t}+\sqrt{2}}.
\end{aligned}$$

Thus

$$\begin{aligned}\lim_{t\to 0}\frac{\sqrt{2-t}-\sqrt{2}}{t} &= \lim_{t\to 0}\frac{-1}{\sqrt{2-t}+\sqrt{2}}\\ &= \frac{-1}{\lim_{t\to 0}\left(\sqrt{2-t}+\sqrt{2}\right)}\\ &= \frac{-1}{\sqrt{2}+\sqrt{2}} = \frac{-1}{2\sqrt{2}}.\end{aligned}$$

(c) $\lim_{x\to\infty}\frac{3x^3+x-5}{4x^3+x^2+6}$.

Solution: For $x \neq 0$, we have

$$\frac{3x^3+x-5}{4x^3+x^2+6} = \frac{3+\frac{1}{x^2}-\frac{5}{x^3}}{4+\frac{1}{x}+\frac{6}{x^3}}.$$

Therefore,

$$\begin{aligned}\lim_{x\to\infty}\frac{3x^3+x-5}{4x^3+x^2+6} &= \lim_{x\to\infty}\frac{3+\frac{1}{x^2}-\frac{5}{x^3}}{4+\frac{1}{x}+\frac{6}{x^3}}\\ &= \frac{\lim_{x\to\infty}\left(3+\frac{1}{x^2}-\frac{5}{x^3}\right)}{\lim_{x\to\infty}\left(4+\frac{1}{x}+\frac{6}{x^3}\right)}\\ &= \frac{3+0+0}{4+0+0} = \frac{3}{4}.\end{aligned}$$

(d) $\lim_{x\to\infty}\left(\sqrt{9x^2+x+2}-3x\right)$.

Solution: For $x > 0$, we have

$$\begin{aligned}\sqrt{9x^2+x+2}-3x &= \left(\sqrt{9x^2+x+2}-3x\right)\frac{\sqrt{9x^2+x+2}+3x}{\sqrt{9x^2+x+2}+3x}\\ &= \frac{\left(\sqrt{9x^2+x+2}\right)^2-(3x)^2}{\sqrt{9x^2+x+2}+3x}\\ &= \frac{9x^2+x+2-9x^2}{\sqrt{9x^2+x+2}+3x}\\ &= \frac{x+2}{\sqrt{9x^2+x+2}+3x}\\ &= \frac{1+\frac{2}{x}}{\sqrt{9+\frac{1}{x}+\frac{2}{x^2}}+3}.\end{aligned}$$

Hence,

$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{9x^2+x+2}-3x\right) &= \lim_{x\to\infty}\frac{1+\frac{2}{x}}{\sqrt{9+\frac{1}{x}+\frac{2}{x^2}}+3}\\ &= \frac{\lim_{x\to\infty}\left(1+\frac{2}{x}\right)}{\lim_{x\to\infty}\left(\sqrt{9+\frac{1}{x}+\frac{2}{x^2}}+3\right)}\\ &= \frac{1+\lim_{x\to\infty}\frac{2}{x}}{\sqrt{9+\lim_{x\to\infty}\left(\frac{1}{x}+\frac{2}{x^2}\right)}+3}\\ &= \frac{1}{\sqrt{9}+3}=\frac{1}{6}.\end{aligned}$$

4. Find all horizontal and vertical asymptotes of the function

$$f(x)=\frac{x^2+4}{x^2-1}.$$

Solution:

The horizontal asymptotes of $f(x)$ are determined by $\lim_{x\to\pm\infty} f(x)$, if the limits exist. For $f(x)$ as above, we have

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{1+\frac{4}{x^2}}{1-\frac{1}{x^2}}=1.$$

Similarly, $\lim_{x\to-\infty} f(x)=1$. Hence, *the graph has only one horizontal asymptote* $y=1$.

The vertical asymptotes $x=a$ are determined by the numbers a such that $\lim_{x\to a-} f(x)=\pm\infty$ or $\lim_{x\to a+} f(x)=\pm\infty$. This happens for $a=1$ and $a=-1$. Indeed

$$f(x)=\frac{x^2+4}{x^2-1}=\frac{x^2+4}{(x-1)(x+1)}$$

and thus

$$\begin{aligned}\lim_{x\to 1-} f(x) &= -\infty\\ \lim_{x\to 1+} f(x) &= \infty\\ \lim_{x\to -1-} f(x) &= \infty\\ \lim_{x\to -1+} f(x) &= -\infty\end{aligned}$$

We conclude that *the vertical asymptotes are* $x=1$ *and* $x=-1$.

5. Find the equation of the tangent to the graph of the function

$$f(x)=\frac{x}{1+2x}$$

at the point $P(1,1/3)$.

Solution: The slope of the tangent at $P(a, f(a))$ is given by the formula

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

For $f(x)$ as above and $a = 1$, we have $f(a) = 1/3$ and

$$\begin{aligned}
\frac{f(a+h) - f(a)}{h} &= \frac{1}{h}\left[\frac{1+h}{1+2(1+h)} - \frac{1}{3}\right] \\
&= \frac{1}{h}\left[\frac{1+h}{3+2h} - \frac{1}{3}\right] \\
&= \frac{1}{h}\frac{3(1+h) - (3+2h)}{3(3+2h)} \\
&= \frac{1}{h}\frac{h}{3(3+2h)} \\
&= \frac{1}{3(3+2h)}.
\end{aligned}$$

Thus

$$m = \lim_{h \to 0} \frac{1}{3(3+2h)} = \frac{1}{9}.$$

The equation of the tangent is

$$y = mx + (f(a) - ma).$$

Since $a = 1$ and $f(a) = 1/3$ and $m = 1/9$, this results in the equation

$$y = \frac{1}{9}x + \frac{2}{9}.$$