Mathematics 19A; Fall 2001; V. Ginzburg
Practice Midterm II; Solutions

1. For each of the ten questions below, state whether the assertion is *true* or *false*.
   (a) Let $f(x)$ and $g(x)$ be differentiable functions. Then $(f(x)g(x))' = f'(x)g(x) + g'(x)f(x)$.
   Answer: **T**.
   (b) $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$.
   Answer: **T**.
   (c) The function $f(x) = \sqrt[3]{x}$ is differentiable at 0.
   Answer: **F**.
   (d) $\lim_{x\to 0}\frac{\tan(7x)}{x} = 7$.
   Answer: **T**.
   (e) $\frac{d}{dx}\tan x = \frac{1}{\sin^2 x}$.
   Answer: **F**; $(\tan x)' = 1/\cos^2 x$.
   (f) $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$.
   Answer: **T**.
   (g) Every continuous functions is differentiable.
   Answer: **F**; for example, $y = |x|$ is continuous but not differentiable at $x = 0$.
   (h) $\frac{d}{dx}\ln|x| = \frac{1}{x}$.
   Answer: **T**.
   (i) $\frac{d}{dx}a^x = ax^{a-1}$.
   Answer: **F**; $\frac{d}{dx}a^x = (\ln a)a^x$.
   (j) $\lim_{x\to 0}\frac{\sin x}{x} = 1$.
   Answer: **T**.

2. Find $f'(x)$ for the following functions.
   (a) $f(x) = \tan(\cos x)$. Solution: By the chain rule:
   $$f'(x) = \frac{1}{\cos^2(\cos x)}(\cos x)' = \frac{-\sin x}{\cos^2(\cos x)}.$$
   (b) $f(x) = \log_2(\sin^{-1}x)$. Solution: By the chain rule:
   $$f'(x) = \frac{1}{(\ln 2)\sin^{-1}x}\left(\sin^{-1}x\right)' = \frac{1}{(\ln 2)\sin^{-1}x}\cdot\frac{1}{\sqrt{1-x^2}} = \frac{1}{(\ln 2)(\sin^{-1}x)\sqrt{1-x^2}}.$$
   (c)
   $$f(x) = \frac{x\ln x}{e^x}.$$
   Solution: Using the quotient rule,
   $$\begin{aligned} f'(x) &= \frac{(x\ln x)'e^x - (x\ln x)(e^x)'}{(e^x)^2} \\ &= \frac{(\ln x + 1)e^x - x\ln x(e^x)}{(e^x)^2} \\ &= \frac{(\ln x + 1) - x\ln x}{e^x} \\ &= \frac{1 + (1-x)\ln x}{e^x}. \end{aligned}$$

(d)

$$f(x) = \frac{\sqrt{1+x}}{(x^3-5)^7}.$$

Solution: By the method of logarithmic differentiation,

$$g'(x) = g(x)\frac{d}{dx}[\ln g(x)].$$

Take

$$g(x) = \frac{\sqrt{1+x}}{(x^3-5)^7}.$$

Then

$$\ln g(x) = \frac{1}{2}\ln(1+x) - 7\ln(x^3-5)$$

and

$$\frac{d}{dx}[\ln g(x)] = \frac{1}{2(1+x)} - \frac{21x^2}{x^3-5}.$$

Thus

$$f'(x) = \frac{\sqrt{1+x}}{(x^3-5)^7}\left(\frac{1}{2(1+x)} - \frac{21x^2}{x^3-5}\right).$$

Alternatively, one can use the quotient rule.

3. Let $f(x) = \frac{1-x^2}{1+x^2}$.

(a) Find $f'(x)$.
Solution: By the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(1-x^2)'(1+x^2) - (1-x^2)(1+x^2)'}{(1+x^2)^2} \\ &= \frac{(-2x)(1+x^2) - (1-x^2)(2x)}{(1+x^2)^2} \\ &= \frac{-2x - 2x^3 - 2x + 2x^3}{(1+x^2)^2} \\ &= \frac{-4x}{(1+x^2)^2}. \end{aligned}$$

(b) Find the equation of the tangent line to the graph of $f(x)$ at the point $P(1,0)$.

Solution: The equation of the tangent to the graph of $y = f(x)$ at $P(a, f(a))$ is $y = mx + f(a) - ma$, where $m = f'(a)$. For $f(x)$ as above, we have $m = f'(1) = -4/2^2 = -1$. Thus, the equation is

$$y = (-1)x + 0 - (-1)\cdot 1 = -x + 1.$$

4. Find $\frac{dy}{dx}$ by implicit differentiation, where $y = f(x)$ is given by the equation

$$x^2y + y^2x = 2x$$

and $x = 1$ and $y = -2$.
Solution: By the method of implicit differentiation we have:

$$\begin{aligned}
\frac{d}{dx}\left(x^2y + y^2x\right) &= \frac{d}{dx}(2x). \\
\frac{d}{dx}\left(x^2y\right) + \frac{d}{dx}\left(y^2x\right) &= 2. \\
2xy + x^2\frac{dy}{dx} + 2y\frac{dy}{dx}x + y^2 &= 2. \\
x^2\frac{dy}{dx} + 2y\frac{dy}{dx}x &= 2 - (2xy + y^2). \\
(x^2 + 2yx)\frac{dy}{dx} &= 2 - 2xy - y^2.
\end{aligned}$$

Solving the last equation for dy/dx, we obtain

$$\frac{dy}{dx} = \frac{2 - 2xy - y^2}{x^2 + 2yx}$$

Finally, we plug in $x = 1$ and $y = -2$:

$$\frac{dy}{dx} = \frac{2 + 4 - 4}{1 - 4} = -\frac{2}{3}.$$

Remark: Alternatively, we can plug $x = 1$ and $y = -2$ in the third equation $2xy + x^2\frac{dy}{dx} + 2y\frac{dy}{dx}x + y^2 = 2$ and then solve the resulting equation for dy/dx.