Mathematics 19A; Fall 2001; V. Ginzburg
Practice Final Solutions

1. For each of the ten questions below, state whether the assertion is *true* or *false*.

(a) Let $f(x)$ be continuous at $x = a$. Then $\lim_{x\to a} f(x) = f(a)$. *Answer:* **T.**

(b) Let f be a differentiable function and $f'(c) = 0$. Then $f(x)$ necessarily has a local maximum or a local minimum at $x = c$. *Answer:* **F:** Counterexample: $f(x) = x^3$, $c = 0$.

(c) Let $f(x) = a^x$. Then $f'(x) = xa^{x-1}$. *Answer:* **F:** $f'(x) = (\ln a)a^x$.

(d) Let $f(x) = \ln|x|$. Then $f'(x) = 1/x$. *Answer:* **T.**

(e)
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)},$$
provided that the limits exist and $\lim_{x\to a} g(x) \neq 0$. *Answer:* **T.**

(f) Assume that $f'(c) = 0$ and $f''(c) > 0$. Then $y = f(x)$ has a local maximum at $x = c$. *Answer:* **F.** The function has a local minimum at $x = c$.

(g) The function
$$f(x) = \begin{cases} x - 2 & \text{for } x < -1, \\ x^2 - 4 & \text{for } x \geq -1. \end{cases}$$
is continuous at $x = -1$. *Answer:* **T.**

(h) The function $f(x) = \sqrt{|x|}$ is differentiable at $x = 0$. *Answer:* **F.**

(i) Assume that $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) and $f(a) = f(b)$. Then there exists a number c in (a, b) such that $f'(c) = 0$. *Answer:* **T.** This is Rolle's theorem.

(j) Let $f(x)$ and $g(x)$ be differentiable functions and $g'(a) \neq 0$. Then, by L'Hospital's rule, one necessarily has that
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$
Answer: **F:** For L'Hospital's rule to apply, $\lim_{x\to a} \frac{f(x)}{g(x)}$ must be an indeterminacy of the form $0/0$ or ∞/∞.

2. Find the following limits

(a) $\lim_{t\to 0}\frac{\sqrt{t+9}-3}{t}$. *Solution:*

$$\begin{aligned}\frac{\sqrt{t+9}-3}{t} &= \frac{\sqrt{t+9}-3}{t}\cdot\frac{\sqrt{t+9}+3}{\sqrt{t+9}+3}\\ &= \frac{(\sqrt{t+9})^2-3^2}{t(\sqrt{t+9}+3)}\\ &= \frac{t+9-9}{t(\sqrt{t+9}+3)}\\ &= \frac{1}{\sqrt{t+9}+3},\end{aligned}$$

for $t\neq 0$. Hence,

$$\lim_{t\to 0}\frac{\sqrt{t+9}-3}{t}=\lim_{t\to 0}\frac{1}{\sqrt{t+9}+3}=\frac{1}{\sqrt{0+9}+3}=\frac{1}{6}.$$

Alternatively, one can use L'Hospital's rule.

(b) $\lim_{x\to\infty}\frac{\ln x}{\sqrt[3]{x}}$. *Solution:* We have $\ln x\to\infty$ and $\sqrt[3]{x}\to\infty$ as $x\to\infty$. Hence, L'Hospital's rule applies. By L'Hospital's rule,

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt[3]{x}}=\lim_{x\to\infty}\frac{(\ln x)'}{(\sqrt[3]{x})'}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{3x^{2/3}}}=\lim_{x\to\infty}\frac{3x^{2/3}}{x}=\lim_{x\to\infty}\frac{3}{\sqrt[3]{x}}=0.$$

(c) $\lim_{x\to 0}\frac{1-\cos x}{x^2}$. We have $1-\cos x\to 0$ and $x^2\to 0$ as $x\to 0$. Hence, L'Hospital's rule applies. By L'Hospital's rule,

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{(1-\cos x)'}{(x^2)'}=\lim_{x\to 0}\frac{\sin x}{2x}.$$

Now we apply L'Hospital's rule again:

$$\lim_{x\to 0}\frac{\sin x}{2x}=\lim_{x\to 0}\frac{(\sin x)'}{(2x)'}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{\cos 0}{2}=\frac{1}{2}.$$

(d) $\lim_{x\to 1-}(1-x)\tanh^{-1}x$. *Solution:* This is an indeterminacy of the type $0\cdot\infty$. Write it as

$$\lim_{x\to 1-}(1-x)\tanh^{-1}x=\lim_{x\to 1-}\frac{\tanh^{-1}x}{(1-x)^{-1}},$$

which is an indeterminacy of the type $0/0$, and hence L'Hospital's rule applies.

Thus,

$$\begin{aligned}
\lim_{x\to 1-}\frac{\tanh^{-1}x}{(1-x)^{-1}} &= \lim_{x\to 1-}\frac{[\tanh^{-1}x]'}{[(1-x)^{-1}]'}\\
&= \lim_{x\to 1-}\frac{(1-x^2)^{-1}}{(1-x)^{-2}}\\
&= \lim_{x\to 1-}\frac{(1-x)^2}{1-x^2}\\
&= \lim_{x\to 1-}\frac{1-x}{1+x}=\frac{1-1}{1+1}=0.
\end{aligned}$$

3. Find $f'(x)$ for the following functions.

(a) $f(x)=\frac{x^2}{1+x^2}$. *Solution:*

$$\begin{aligned}
f'(x) &= \frac{(x^2)'(1+x^2)-x^2(1+x^2)'}{(1+x^2)^2}=\frac{2x(1+x^2)-x^2(2x)}{(1+x^2)^2}\\
&= \frac{2x+2x^3-2x^3}{(1+x^2)^2}=\frac{2x}{(1+x^2)^2}.
\end{aligned}$$

(b) $f(x)=\sin\left(\frac{\ln x}{x}\right)$. *Solution:* By the chain rule:

$$f'(x)=\cos\left(\frac{\ln x}{x}\right)\cdot\left(\frac{\ln x}{x}\right)'.$$

By the quotient rule:

$$\begin{aligned}
\left(\frac{\ln x}{x}\right)' &= \frac{(\ln x)'x-(x)'\ln x}{x^2}\\
&= \frac{x\frac{1}{x}-\ln x}{x^2}\\
&= \frac{1-\ln x}{x^2}.
\end{aligned}$$

Thus,

$$f'(x)=\cos\left(\frac{\ln x}{x}\right)\cdot\frac{1-\ln x}{x^2}.$$

(c) $f(x)=\frac{(x+1)^2}{\sqrt{x^2+2x}}$. *Solution:* Use logarithmic differentiation:

$$f(x)=\frac{(x+1)^2}{\sqrt{x}\sqrt{x+2}}$$

and so

$$\ln f(x)=2\ln(x+1)-\frac{1}{2}\ln x-\frac{1}{2}\ln(x+2).$$

Thus

$$[\ln f(x)]' = \frac{2}{x+1} - \frac{1}{2x} - \frac{1}{2(x+2)}$$

and

$$f'(x) = f(x)[\ln f(x)]' = \frac{(x+1)^2}{\sqrt{x^2+2x}} \left(\frac{2}{x+1} - \frac{1}{2x} - \frac{1}{2(x+2)} \right).$$

(d) $f(x) = x^{-1} \tan^{-1} x^2$. *Solution:* By the product rule:

$$f'(x) = \left(x^{-1}\right)' \tan^{-1} x^2 + x^{-1} \left(\tan^{-1} x^2\right)' = \frac{-\tan^{-1} x^2}{x^2} + x^{-1} \left(\tan^{-1} x^2\right)'.$$

Furthermore, by the chain rule,

$$\left(\tan^{-1} x^2\right)' = \frac{1}{1+(x^2)^2} \cdot \left(x^2\right)' = \frac{2x}{1+x^4}.$$

Thus,

$$f'(x) = -\frac{\tan^{-1} x^2}{x^2} + \frac{2}{1+x^4}.$$

4. Find the equation of the tangent line to the curve $x^2 + xy - y^2 = 1$ at the point $(2, 3)$. *Solution:*

Step 1. Using implicit differentiation,

$$\begin{aligned} \frac{d}{dx}x^2 + \frac{d}{dx}(xy) - \frac{d}{dx}y^2 &= 0, \\ 2x + y + x\frac{dy}{dx} - 2y\frac{dy}{dx} &= 0. \end{aligned}$$

Now let us plug in the values $x = 2$ and $y = 3$. We get:

$$4 + 3 + 2\frac{dy}{dx} - 6\frac{dy}{dx} = 7 - 4\frac{dy}{dx} = 0.$$

Thus $\frac{dy}{dx} = 7/4$. This is the slope m of the tangent.

Step 2. The equation of the tangent through the point (a, b) is $y = mx + b - ma$. Plugging $a = 2$ and $b = 3$, we obtain

$$y = \frac{7x}{4} + 3 - 2 \cdot \frac{7}{4} = \frac{7x}{4} - \frac{1}{2}.$$

5. Let $f(x) = 2x^3 + 3x^2 - 12x + 7$.

(a) Find $f'(x)$ and $f''(x)$. *Solution:*

$$f'(x) = 6x^2 + 6x - 12$$

and

$$f''(x) = 12x + 6.$$

For what follows it's also useful to note that $f'(x) = 6(x-1)(x+2)$.

(b) Find the local maxima and minima of f. *Solution:* Local maxima and minima may occur only when $f'(x) = 0$, i.e.,

$$f'(x) = 6x^2 + 6x - 12 = 0$$

which has solutions -2 and 1. Then $f''(-2) = -18 < 0$ and $f''(1) = 18 > 0$. By the second derivative test, *f has a local maximum at $x = -2$ equal to $f(-2) = 27$ and a local minimum at $x = 1$ equal to $f(1) = 0$.*

(c) Find the intervals of increase and decrease for f. *Solution:* The intervals of increase are given by the inequality

$$f'(x) = 6x^2 + 6x - 12 > 0.$$

Hence, *the intervals of increase are $(\infty, -2)$ and $(1, \infty)$.*
The intervals of decrease are given by the inequality

$$f'(x) = 6x^2 + 6x - 12 < 0.$$

Therefore, *the interval of decrease is $(-2, 1)$.*

(d) Find the inflection points of f. *Solution:* The inflection points are given by the equation $f''(x) = 0$:

$$f''(x) = 12x + 6 = 0.$$

This equation has one solution $-1/2$. Thus *the inflection point is $-1/2$.*

(e) Find the intervals of concavity of f. *Solution:* The intervals of concavity upward are given by the inequality

$$f''(x) = 12x + 6 > 0.$$

Hence, *the interval of concavity upward is $(-1/2, \infty)$.*
The intervals of concavity downward are given by the inequality

$$f''(x) = 12x + 6 < 0.$$

Hence, *the interval of concavity downward is $(-\infty, -1/2)$.*

6. Find the absolute maximum and the absolute minimum of $f(x) = xe^{-x^2/2}$ on $[0, 2]$.

Solution: Let us first find the critical numbers of $f(x)$, i.e., solutions to $f'(x) = 0$:

$$\begin{aligned} f'(x) &= (x)'e^{-x^2/2} + x\left(e^{-x^2/2}\right)' = e^{-x^2/2} + xe^{-x^2/2}\left(-x^2/2\right)' \\ &= e^{-x^2/2} - x^2e^{-x^2/2} = e^{-x^2/2}(1 - x^2). \end{aligned}$$

The solutions to

$$f'(x) = e^{-x^2/2}(1 - x^2) = 0 \quad \text{or, equivalently,} \quad 1 - x^2 = 0$$

are $x = \pm 1$. Only one of these numbers, namely $x = 1$, is in the interval $[0, 2]$.

To find the absolute maximum and the absolute minimum we need to pick the greatest and the smallest value of $f(x)$ among its values at the end points of the interval and the critical numbers within the interval:

$$\begin{aligned} f(0) &= 0, \\ f(1) &= e^{-1/2}, \quad \text{(critical number)} \\ f(2) &= 2e^{-2}. \end{aligned}$$

Note that $0 < 2e^{-2} < e^{-1/2}$. Thus the absolute maximum is $f(1) = e^{-1/2}$ and the absolute minimum is $f(0) = 0$.

7. Find the area of the largest rectangle that can be inscribed in a right triangle with legs of length a and b if two sides of the rectangle lie along the legs.

   Solution: Let x be the length of the side along the leg of length a and y be the length of the side along the leg of length b.

   Then

$$\frac{b-y}{x} = \frac{b}{a}.$$

   Solving this equation for y, we obtain

$$y = b\left(1 - \frac{x}{a}\right).$$

   The area of the rectangle is

$$A = xy = bx\left(1 - \frac{x}{a}\right).$$

   Thus

$$\frac{dA}{dx} = b\left(1 - \frac{x}{a}\right) + bx\left(-\frac{1}{a}\right) = b\left(1 - \frac{2x}{a}\right).$$

   Solving

$$\frac{dA}{dx} = b\left(1 - \frac{2x}{a}\right) = 0,$$

we obtain $x = a/2$, which corresponds to the area $A = ab/4$. As in the previous problem $x = a/2$ is a critical number and to find the maximal area we need to compare $A = ab/4$ with A for the extreme values of x which are 0 and a. For both of these values of x, we have $A = 0$. Hence $A = ab/4$ is the largest possible area.