Mathematics 103A; Winter 2016; V. Ginzburg
Practice Midterm

1. For each of the ten questions below, state whether the assertion is *true* or *false*. (You do not need to justify your answer.)

   (a) The subset of $\mathbb{C}$ given by the conditions $x \leq y^2$ is closed.

   (b) $\operatorname{Log} z$ is an entire function.

   (c) Every analytic function has an antiderivative in its domain.

   (d) $|\sin z| \leq 1$ for all $z \in \mathbb{C}$.

   (e) Let C be the circle of radius 1 centered at the origin. Then

   $$\int_C f(z)\,dz = 0$$

   for every function analytic in the disk of radius 2 centered at 0.

   (f) The function $f(z) = z\bar{z}$ is analytic.

   (g) The equation $z^n = w$ has exactly n solutions when $w \neq 0$.

   (h) $\lim_{z\to\infty} \frac{\bar{z}}{z} = 1$.

   (i) Every analytic function is necessarily continuous.

   (j) The Cauchy–Goursat theorem asserts that if f is analytic at all points interior to and on a simple closed contour C, then

   $$\int_C f(z)\,dz = 0.$$

2. Find all solutions of the following equations:

   (a) $z^4 = -8 - 8\sqrt{3}i$,

   (b) $e^{2iz} = -1 + \sqrt{3}i$.

3. Use the Cauchy–Riemann equations to determine which of the following functions f are analytic and, when this is the case, find the derivative $f'(z)$ expressed as a function of z:

   (a) $f(z) = x^2 + y^2 + 2xyi$,

   (b) $f(z) = z \operatorname{Im} z$,

   (c) $f(z) = e^{-\theta}\big(\cos(\ln r) + i\sin(\ln r)\big)$.

4. Evaluate the following integrals:

   (a) $\int_C \frac{dz}{z^n}$, where C is the upper half-circle $z(\theta) = Re^{i\theta}$, $0 \leq \theta \leq \pi$,

   (b) $\int_i^{i+1} e^{\pi z}\, dz$,

   (c) $\int_C e^{\bar{z}}\, dz$, where C is the triangle with vertices 1, $1+i$ and i oriented counterclockwise.

5. (a) Let f be an entire function such that $\operatorname{Re} f(z)$ is constant. Prove that f is constant.

   (b) Let f be a function analytic in a connected domain and $\operatorname{Arg} f(z)$ is constant. Prove that f is constant.

6. Let C be the upper half-circle $|z| = 2$. Prove that

$$\left| \int_C \frac{e^{z/2}(z-2)}{z^4+1}\, dz \right| \leq \frac{8\pi e}{15}.$$