Mathematics 103A; Winter 2016; V. Ginzburg
Practice Final

1. For each of the ten questions below, state whether the assertion is *true* or *false*. (You do not need to justify your answer.)
   (a) The domain $D = \{z \in \mathbb{C} \colon |\operatorname{Im} z| > 1\}$ is simply connected.
   (b) The equation $z^7 = 8$ has exactly 8 distinct solutions.
   (c) The function $\frac{\cos z - 1}{(1-z^5)z^2}$ has a pole of order 5 at $z_0 = 0$.
   (d) $\operatorname{Log}(-1) = \pi$.
   (e) The Taylor series of the function $f(z) = \frac{e^z}{(z-10)(z-i)}$ at $z_0 = 1$ has radius of convergence $\sqrt{2}$.
   (f) The function $f = u + iv$ is analytic at a point z if and only if $u_x = v_y$ and $u_y = -v_x$ at z.
   (g) Assume that f is analytic and bounded on the domain $0 < |z| < \epsilon$, but not at 0. Then f has a removable singularity at 0.
   (h) Assume that f has an essential singularity at z_0. Then $\lim_{z \to z_0} f(z) = \infty$.
   (i) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges conditionally.
   (j) The power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} z^n$ converges uniformly in the disk $|z| \le 1/2$.

2. Find the Taylor series of the following functions:
   (a) $f(z) = z\cosh(z^2)$ at $z_0 = 0$,
   (b) $f(z) = \dfrac{z}{z^4 + 4}$ at $z_0 = 0$,
   (c) $f(z) = 1/e^z$ at $z_0 = 1$.

3. Find the Laurent series expansions of the following functions:
   (a) $f(z) = \dfrac{1}{z^2(1-z)}$ in the region $0 < |z| < 1$,
   (b) $f(z) = \dfrac{\sinh z}{z^2}$ in the region $z \neq 0$,
   (c) $f(z) = z^2 e^{1/z}$ in the region $z \neq 0$.

4. Find and classify the singularities of the following functions f. If a singularity is a pole, find its order. Find the residues at these singularities.

   (a) $$f(z) = \frac{e^z}{z^2 + \pi^2},$$

   (b) $$f(z) = \frac{1}{z} \cos \frac{1}{z-1},$$

   (c) $$f(z) = \frac{1}{z(e^z - 1)}.$$

5. Use Cauchy's residue theorem to evaluate the integrals

   (a) $\displaystyle\int_C z^2 e^{1/z}\, dz$, where C is the circle $|z| = 5$ oriented counterclockwise.

   (b) $\displaystyle\int_C \frac{4z-5}{z(z-2)}\, dz$, where C is the square $|x| + |y| = 1$ oriented counterclockwise.

6. Use residues to evaluate the integrals

   (a) $$\int_0^\infty \frac{x^2\, dx}{x^6 + 1},$$

   (b) $$\int_{-\infty}^\infty \frac{\cos x\, dx}{x^2 + 1}.$$

7. Prove that there exist no analytic function f on the unit disk such that $f(1/n) = \sin(1/n)$ and $f(-1/n) = 1/n$ for all positive integers $n \geq 1$.

8. Prove that there exist no analytic function f on the closed disk $|z| \leq 2$ such that $f(z) = 0$ on the boundary of the disk while $f(1) = i$.

9. Let f be an entire function such that $|10 - f(z)| < 1$ for all z. Prove that f is constant.