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# RELATIVE ALMOST EXISTENCE THEOREM AND THE HAMILTONIAN SEIFERT CONJECTURE

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**Abstract**

Relative Almost Existence Theorem and the Hamiltonian Seifert Conjecture

by

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The main theme of this thesis is the existence of periodic orbits for Hamiltonian dynamical systems.

We prove the relative almost existence theorem asserting that almost all low levels of a function on a symplectic manifold carry contractible periodic orbits of the Hamiltonian flow, provided that the function attains its minimum along a closed symplectic submanifold and the symplectic manifold meets some natural requirements. As an immediate consequence, we obtain the existence of contractible periodic orbits on almost all low energy levels for systems describing the motion of a charge in a symplectic magnetic field.

In general, periodic orbits need not exist on all low levels even when the phase space is four dimensional. We construct a proper $C^2$-smooth function on $\mathbb{R}^4$ such that its Hamiltonian flow has no periodic orbits on at least one regular level set. This result can be viewed as a $C^2$-smooth counterexample to the Hamiltonian Seifert conjecture in dimension four. We also give examples of functions with a sequence of regular levels without periodic orbits, converging to an isolated, but very degenerate, minimum.

The proof of the relative almost existence theorem hinges on the notion of the relative Hofer–Zehnder capacity and on showing that this capacity of a small neighborhood of a symplectic submanifold is finite. The latter is carried out by proving that the Floer homology of a test function is non-zero for a suitable interval of actions.

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# Chapter 1

# Introduction

The problem of existence of periodic orbits on a given level of an autonomous Hamiltonian is one of the central questions addressed by symplectic topology. In this context, Viterbo's proof of the Weinstein conjecture, [Vi1], and its refinement, the almost existence theorem, are among the most important results. The almost existence theorem, proved by H. Hofer and E. Zehnder, [HZ1, HZ2], and by M. Struwe, [St], asserts that almost all (in the sense of measure theory) regular levels of a proper $C^2$-smooth function on $\mathbb{R}^{2n}$ carry periodic orbits of the Hamiltonian flow. In the last decade, these theorems have been extended to a broad class of symplectic manifolds; see, e.g., [FHV, HV1, HV2, LT, Lu1, Ma]. However, as has been noted by E. Zehnder, the almost existence theorem fails for a general symplectic manifold (see [HZ2, Ze]).

In this thesis, we prove a relative version of the almost existence theorem, [GG3]. Namely, consider a function $F$ on a geometrically bounded symplectically aspherical manifold $W$, with its minimum (say, equal to zero) attained along a closed symplectic submanifold $M$. The relative almost existence theorem asserts that the levels $F = \epsilon$ carry contractible periodic orbits of the Hamiltonian flow of $F$ for almost all small $\epsilon > 0$. This result strengthens a theorem of [CGK] guaranteeing the existence of periodic orbits for a dense set of values of $F$ near zero.

This investigation has been inspired by the existence problem for periodic orbits of twisted geodesic flows which describe the motion of a charge in a magnetic field (see Section 2.1.3 for definitions). As an immediate consequence of the relative almost existence theorem, we obtain the existence of contractible twisted geodesics (i.e., trajectories of a charge in a magnetic field) on almost all low energy levels, provided that the magnetic field is symplectic – a theorem strengthening or complementing numerous other results on the existence of twisted geodesics; see, e.g., [CGK, FS1, FS2, Gi2, GK1, GK2, Ke1, Lu1, Mac1, Pol2].

Symplectic topology offers a powerful technique – Floer homology – for finding periodic orbits of a Hamiltonian flow with fixed period (say, equal to one). This technique uses the least action principle to identify one-periodic orbits as critical points of the action functional. The existence of critical points is then proved by calculating a version of Morse homology (called Floer homology) for the action functional. However, this technique does not apply to the problem of existence of periodic orbits on a given level or a set of levels, for the periods of these orbits are unknown. Moreover, when applied directly to a fixed energy level, the least action principle leads to an intractable problem of showing that the critical points of the action functional exist. The difficulty lies in associating a Morse homology theory to the action functional for curves on a given level. Furthermore, the resulting homology would in general be zero since, as we will soon see, periodic orbits need not exist on a fixed level unless the level satisfies some additional conditions.

To circumvent these difficulties, the problem is usually reduced to that of finding periodic orbits of a fixed period. Similarly to the original almost existence theorem and to many other results of this type, the proof of the relative almost existence theorem is comprised of two steps.

The first step is showing that the flow of a Hamiltonian with sufficiently large variation has a non-trivial contractible one-periodic orbit. More precisely, we consider a Hamiltonian $H$ supported in a neighborhood of $M$, constant near $M$, and attaining its maximum along $M$. Then, if the maximum is large enough, the flow has a non-trivial contractible one-periodic orbit. This is proved by showing that the Floer homology of $H$ does not vanish for an interval of actions $[a, b)$ with $\max H < a$ (to ensure that the orbit is non-trivial). Note that establishing the existence of non-trivial one-periodic orbits is a common first step in proving almost existence theorems. For instance, our theorem generalizes a theorem proved in [HZ2]. Similar theorems, under no assumptions on $W$, but with some additional constraints on the normal bundle to $M$ (e.g., that the normal bundle is trivial), have been proved in [HV2, LT, Lu2]. We also prove a (relatively easy) version of this theorem for time-dependent flows along the lines of [BPS]. Here, the orbits are still contractible, but not necessarily non-trivial.

The second step is to introduce the relative Hofer–Zehnder capacity. This capacity is defined as in [HZ2], but for functions constant near $M$. The existence theorem for one-periodic orbits guarantees that the capacity of a small neighborhood of $M$ relative to $M$ is finite. The almost existence theorem follows now, exactly as in [HZ2], from the fact that the capacity of the sets $\{F < \epsilon\}$ is an increasing function of $\epsilon > 0$.

The idea to consider the relative Hofer–Zehnder capacity goes back to [La], and a number of existence results akin to those proved here can be interpreted as calculations of this capacity; see, e.g., [HV2, LT, Lu2]. Certain other versions of relative capacities have been recently introduced, [CGK, BPS, Lu2, Sc], but, to the best of our knowledge, the relative Hofer–Zehnder capacity has not been put to systematic use till now. It should be noted that it is not known whether or not this capacity differs from the original Hofer–Zehnder capacity for neighborhoods of symplectic submanifolds (see Example 2.1.11 and Section 3.3.3). Yet, in this setting, we can only establish finiteness of the relative capacity, and hence this capacity readily lends itself as a convenient tool for proving almost existence results.

Next, let us address the question whether or not periodic orbits must exist on all regular energy levels near the minimum. We first consider the case when $M$ is the origin in $\mathbb{R}^{2n}$.

Regular levels without periodic orbits of a $C^2$-smooth proper Hamiltonian on $\mathbb{R}^{2n}$, as is manifested by the almost existence theorem (see Remark 2.3.3), are exceptional in the sense of measure theory. (We call such levels *aperiodic.*) Thus, the question is whether aperiodic levels really exist. Here we give an affirmative answer to this question in dimension four. In other words, we construct, closely following [GG1, GG2], a $C^2$-smooth function on $\mathbb{R}^4$ with at least one regular level set having no periodic orbits of the Hamiltonian flow. Following the tradition of [Gi1, Gi3, Gi5, He1, He2, Ke2, KuG, KuGK, KuK1, KuK2, Schwe], this result can be called a $C^2$-smooth counterexample to the Hamiltonian Seifert conjecture in dimension four. More generally, the "Hamiltonian Seifert conjecture" is the question whether or not there exists a proper function on $\mathbb{R}^{2n}$ whose Hamiltonian flow has no periodic orbits on at least one regular level set.

In dimensions greater than six, $C^\infty$-smooth counterexamples to the Hamiltonian Seifert conjecture were constructed by V. Ginzburg, [Gi1], and simultaneously by M. Herman, [He1, He2]. In dimension six, a $C^{2+\epsilon}$-smooth counterexample was found by M. Herman, [He1, He2]. This smoothness constraint was later relaxed to $C^\infty$ by V. Ginzburg, [Gi3]. A very simple and elegant construction of a new $C^\infty$-smooth counterexample in dimensions greater than four was recently discovered by E. Kerman, [Ke2]. The flow in E. Kerman's example has dynamics different from the ones in [Gi1, Gi3, He1, He2]. We refer the reader to [Gi4, Gi5] for a detailed discussion of the Hamiltonian Seifert conjecture. The reader interested in the results concerning the original Seifert conjecture settled by K. Kuperberg, [KuGK, KuK1], should consult [KuK2, KuK3]. Here we only mention that a $C^1$-smooth counterexample to the

Seifert conjecture on $S^3$ was constructed by P. Schweitzer, [Schwe]. Later, the smoothness in this example was improved to $C^2$ by J. Harrison, [Ha]. A $C^1$-smooth volume–preserving counterexample on $S^3$ was found by G. Kuperberg, [KuG]. The ideas from both P. Schweitzer's and G. Kuperberg's constructions play an important role in the construction given in [GG1, GG2], and hence in the one presented here.

The next question is then how large the collection of aperiodic levels can be and if such levels can exist when $M$ is different from a point. When $M$ is a Morse non-degenerate minimum of $F$ as is the case, for instance, for twisted geodesic flows, the flow has, hypothetically, a periodic orbit on every low energy level. Moreover, one can expect a certain lower bound on the number of such periodic orbits in terms of the cohomology and codimension of $M$. This conjecture can be thought of as a plausible generalization of the Weinstein–Moser theorem, [Mo, We]. (We refer the reader to [GK1, Ke1] for a detailed discussion of this conjecture and the proofs of certain particular cases.) The relative almost existence theorem provides further evidence supporting the conjecture.

If the minimum of $F$ along $M$ is not assumed to be Morse–Bott non-degenerate, periodic orbits of the flow need not exist on all levels of $F$ near the minimum. In a variety of settings, we construct Hamiltonians $F$ such that for some sequence of regular values $\epsilon_k \to 0+$ the levels $F = \epsilon_k$ carry no periodic orbits of the flow; see Section 2.3.2. In these examples, zero is an isolated, but very degenerate, critical value. Such Hamiltonians also exist on the standard cotangent bundles to spheres. These examples are obtained by combining A. Katok's construction of asymmetric Finsler geodesic flows with finitely many closed geodesics, [Ka, Zi], the elimination of periodic orbits as in [Gi4, GG1, GG2, Ke2], and a suitable smoothing procedure, [Se]; see Sections 2.3.2 and 4.5.

Throughout the thesis, numerous notions and theorems from symplectic topology are often utilized without explanation. The goal of this work has been to present only new and original results, rather than to give an introduction to the subject. We refer the reader to [HZ2, McDS] for a detailed introduction to the methods used here.

# Chapter 2

# Main results

In this chapter we state and discuss the main results of the thesis. In Section 2.1, we state our results concerning the existence problem of periodic orbits for certain autonomous Hamiltonian dynamical systems and define and briefly discuss the relative Hofer–Zehnder capacity. Here we also state the relative almost existence theorem and derive the almost existence of periodic orbits on low energy levels of cotangent bundles twisted by a symplectic magnetic form. The version of the theorem on the existence of one-periodic orbits for time-dependent Hamiltonians is stated in Section 2.2. We give examples of Hamiltonian flows without periodic orbits in Section 2.3.

## 2.1 Contractible periodic orbits of autonomous flows

In this section we are primarily concerned with almost existence theorems. However, our approach requires to first establish the existence of one-periodic orbits for Hamiltonians with sufficiently large variation. Hence, we begin this section by stating some results of this type.

### 2.1.1 Periodic orbits of autonomous Hamiltonian systems

The main theme of this subsection is the general principle asserting that under suitable additional hypotheses, a compactly supported Hamiltonian must have a *fast non-trivial* periodic orbit, provided that the maximum of the Hamiltonian is sufficiently large. (A non-trivial periodic orbit is called *fast* if the orbit has period less than or equal to one; when the period is greater than one, the orbit is called *slow*.) Here, we focus on Hamiltonians supported in a neighborhood of a closed symplectic submanifold $M$ and constrained on $M$. Let us start with a result which holds for a relatively broad class of Hamiltonians.

**Theorem 2.1.1.** *Assume that $M$ is a closed symplectic submanifold of a geometrically bounded symplectically aspherical manifold $W$ (see the definitions below) and $U$ is a sufficiently small neighborhood of $M$. Then there exists a finite constant $C > 0$, depending on $U$, such that for every smooth function $H$ supported in $U$ with $\min_M H > C$, the Hamiltonian flow of $H$ has a non-trivial contractible fast periodic orbit.*

*Remark* 2.1.2. When $W = M \times \mathbb{R}^{2n}$, equipped with the product symplectic structure, the neighborhood $U$ can be taken arbitrarily large but bounded. For $W = \mathbb{R}^{2n}$, Theorem 2.1.1 is due to H. Hofer and E. Zehnder; see [HZ2, Theorem 12, p. 183], where an explicit upper bound on the period is given in terms of the capacity of $U$. A similar upper bound exists in terms of the relative capacity of $(U, M)$; see Remark 2.1.18.

Note also that it is not known whether or not the function $H$ in Theorem 2.1.1 must have a contractible periodic orbit of period exactly equal to one.

Theorem 2.1.1, proved in Section 3.3, is a soft consequence of a result which, under more restrictive assumptions on $H$, allows one to control the actions on periodic orbits and thus distinguish trivial and non-trivial orbits and establish the existence of one-periodic orbits. To state this result, we need first to recall some definitions (including those used in Theorem

2.1.1) and introduce necessary notations.

For a compact subset $Z$ of a symplectic manifold $(V, \omega)$ without boundary, denote by $\mathcal{H}(V, Z)$ the class of smooth functions $H: V \to \mathbb{R}$ such that

(H0) $H$ vanishes on a non-empty open set (depending on $H$) whose complement is compact;

(H1) $H$ is constant on a neighborhood of $Z$; and

(H2) $\max_V H = H|_Z$.

When $V$ is not compact, the condition (H0) is equivalent to requiring $H$ to have compact support. For a compact $V$, this condition is equivalent to that $H$ vanishes on a neighborhood of a point.

If $V$ is a manifold with boundary and $Z$ is disjoint from the boundary $\partial V$, we set $\mathcal{H}(V, Z) = \mathcal{H}(V \smallsetminus \partial V, Z)$. Note that then every function $H \in \mathcal{H}(V, Z)$ extends to a smooth function on $V$ vanishing near $\partial V$.

Let $\omega|_{\pi_2(V)} = 0$. Recall that for a time-dependent Hamiltonian $H: S^1 \times V \to \mathbb{R}$ the action functional on the space of smooth contractible loops is then defined as

$$A_H(x) = -\int_{D^2} \bar{x}^* \omega + \int_{S^1} H(t, x)\, dt, \tag{2.1.1}$$

where $x: S^1 \to V$ is a contractible loop and $\bar{x}: D^2 \to V$ is a map of a disk, bounded by $x$.

A smooth compactly supported function $H$ on an open subset $U$ of $V$ will always be regarded as a smooth compactly supported function on $V$ by extending it as zero to $V \smallsetminus U$. In particular, $\mathcal{H}(U, Z) \subset C^\infty(V)$ when $Z \subset U$ and, for $H \in \mathcal{H}(U, Z)$, the action functional $A_H$ is defined on all contractible loops $x$ in $V$.

The main tool utilized in this thesis to prove the existence of periodic orbits of Hamiltonian flows is Floer homology. To have the Floer homology of $H$ defined, we need to impose some additional conditions on the ambient manifold $V$, which will guarantee that the compactness theorem, (see, e.g., [HZ2]), holds for $V$. The first of these conditions is that $(V, \omega)$ is *symplectically aspherical*, i.e.,

$$\omega|_{\pi_2(V)} = 0 \quad \text{and} \quad c_1(TV)|_{\pi_2(V)} = 0.$$

Furthermore, since $V$ is not assumed to be compact, we need a way to control the geometry of $V$ at infinity. One standard way to do this is to require the manifold to be convex at infinity. However, this requirement is not met in general by twisted cotangent bundles, which serve as one of the motivating examples for this investigation. Hence, we impose a slightly weaker condition and require the manifold to be geometrically bounded.

We refer the reader to Chapters V (by J.-C. Sikorav) and X (by M. Audin, F. Lalonde, and L. Polterovich) in [AL] and to [CGK, Lu1] for a discussion of the concept of geometrically bounded manifolds. Here we only mention that among these manifolds are compact manifolds, manifolds convex at infinity (e.g., $\mathbb{C}^n$), and twisted cotangent bundles.

In what follows, we will always denote a geometrically bounded symplectically aspherical manifold by $W$, while $V$ will stand for a general symplectic manifold. Likewise, $M$ will denote a closed (symplectic, in many instances) submanifold of $W$ or $V$ and $Z$ will be just a compact subset.

Now we are in a position to state the main technical result of this section which is the key to the proof of Theorem 2.1.1 and to the relative almost existence theorem discussed later.

**Theorem 2.1.3.** *Assume that $M$ is a closed symplectic submanifold of a geometrically bounded symplectically aspherical manifold $W$ and $U$ is a sufficiently small neighborhood of $M$. Then there exists a finite constant $C > 0$, depending on $U$, such that for every $H \in \mathcal{H}(U, M)$ with $\max H > C$ the Hamiltonian flow of $H$ has a non-trivial contractible one-periodic orbit with action greater than $\max H$.*

This theorem will be proved in Section 3.2.

*Remark* 2.1.4. When $W = M \times \mathbb{R}^{2n}$, equipped with the product symplectic structure, in Theorem 2.1.3, as in Theorem 2.1.1, the neighborhood $U$ can be taken arbitrarily large but bounded. The proof of Theorem 2.1.3 gives also an upper bound on the value of the constant $C$ (see Theorem 3.2.1). Namely, it is sufficient to take $C = \pi R^2$, where $R$ is the radius of a symplectic tubular neighborhood of $M$ containing $U$; see Section 3.1.1 for the definition. Note also that the constant $C$ in Theorem 2.1.1 can be taken the same as in Theorem 2.1.3. Furthermore, the function $H$ from Theorem 2.1.3 must have non-trivial periodic orbits for every period $T \geq 1$. (Indeed, the orbits of period $T \geq 1$ for $H$ are exactly one-periodic orbits for the rescaled function $T \cdot H$ which clearly satisfies the hypotheses of the theorem.)

*Example* 2.1.5. Let $W = \mathbb{R}^{2n}$ be equipped with the standard symplectic structure. Then $M$ is necessarily a point. In this case, Theorem 2.1.3 is established by H. Hofer and E. Zehnder in [HZ2] under the additional assumption that $H$ is non-negative.

When the normal bundle to $M$ in $W$ is trivial, a result similar to Theorem 2.1.3 was proved by H. Hofer and C. Viterbo by a different method, [HV2]. In [HV2], the manifold $W$ need not be symplectically aspherical, but the function $H$ is required to be non-negative.

*Remark* 2.1.6. In Theorem 2.1.3 and in the almost existence theorems stated below, the assumptions that $W$ is geometrically bounded and symplectically aspherical appear to be superfluous. When $H \geq 0$, it should be possible to prove a version of the theorem without these assumptions, e.g., by utilizing the methods of [HV2, LT, Lu1]. (Some preliminary results in this direction have been obtained by E. Kerman, [Ke3], and L. Macarini, [Mac2].) However, we feel that the Floer homology proof given here is of interest by itself (even for $W = \mathbb{R}^{2n}$, at least because it allows one to eliminate the assumption $H \geq 0$) and this argument may have some other applications.

As stated, Theorem 2.1.3 does not hold when the assumption (H1) that $H$ is constant near $M$ is replaced by the less restrictive assumption that $H$ is constant on $M$; see Example 3.2.4. However, the hypotheses of Theorem 2.1.3 can be replaced by the weaker condition $\min_M H > C$, as Theorem 2.1.1 indicates, at the cost of losing control of the value of the action (and, to some extent, of the period of the orbit). Then, the flow still has a non-trivial contractible periodic orbit with period not exceeding one, but the action on this orbit does not have to be greater than or equal to $\max H$.

Furthermore, as will be clear from the proof, the assumption (H1) can be replaced by the assumption that all partial derivatives of $H$ at $M$ vanish up to fourth order. One can also allow $H$ to be time-dependent. In this case, we need to require $H_t$, $t \in [0,1]$, to be periodic in time and belong to the class $\mathcal{H}(U, M)$ for every $t$, and the maximum of $H$ has to be independent of time. It is not clear whether or not this last condition can be relaxed.

Finally note that Theorem 2.1.3 does not extend to arbitrarily large bounded neighborhoods of $M$ if periodic orbits are still required to be contractible; see Example 2.1.20.

### 2.1.2 Relative Hofer–Zehnder capacity and almost existence

A relative version of the Hofer–Zehnder capacity can be defined in a variety of ways depending on whether or not (and how) the homotopy class of an orbit is incorporated into the definition.

#### Relative Hofer–Zehnder capacity

The simplest definition of the capacity imposes no requirement on the homotopy class of periodic orbits. Thus, let $(V, \omega)$ be a symplectic manifold. The following definition, [La], is the key to deriving the almost existence theorem from Theorem 2.1.3.

**Definition 2.1.7.** Let $Z$ be an arbitrary compact non-empty subset of $V$. The *relative Hofer–Zehnder capacity* $\mathrm{c_{HZ}}(V, Z) \in (0, \infty]$ is defined as

$$\mathrm{c_{HZ}}(V, Z) = \sup_{H \in \mathcal{H}(V,Z)} \{\max H \mid \text{all non-trivial periodic orbits of } H \text{ are slow}\}.$$

When the dependence of this capacity on $\omega$ is essential, we will use the notation $\mathrm{c_{HZ}}(V, Z, \omega)$.

Thus, a function $H \in \mathcal{H}(V, Z)$ with $\max H > \mathrm{c_{HZ}}(V, Z)$ must have a non-trivial fast periodic orbit.

As we have pointed out above, the assumptions (H1) and (H2) in the definition of the class $\mathcal{H}(V, Z)$ can be replaced by the weaker conditions. Namely, for $C > 0$, set

$$\tilde{\mathcal{H}}_C(V, Z) = \{H \in C^\infty(V) \mid H \text{ satisfies (H0) and } \min_Z H > C\}.$$

**Theorem 2.1.8.**

1. $\mathrm{c_{HZ}}(V, Z) = \inf C$, *where the infimum is taken over* $C > 0$ *such that every* $H \in \tilde{\mathcal{H}}_C(V, Z)$ *has a non-trivial fast periodic orbit.*

2. *The capacity* $\mathrm{c_{HZ}}(V, Z)$ *does not change when functions in* $\mathcal{H}(V, Z)$ *or* $\tilde{\mathcal{H}}_C(V, Z)$ *are required to be non-negative.*

The proof of this theorem, which is nearly identical to the proof of Theorem 2.1.1, will be given in Section 3.3.

*Corollary* 2.1.9. $\mathrm{c_{HZ}}(V, \mathit{point})$ is equal to $\mathrm{c_{HZ}}(V)$, the standard Hofer–Zehnder capacity of $V$ (see [HZ2] for the definition).

The main point of this corollary is that the conditions on the function $H$ in the definition of the standard Hofer–Zehnder capacity can be relaxed, [HZ2, p. 70]. Namely, it is not necessary to assume that $H$ vanishes on an open set (rather at one point) or that $H$ attains its maximal value on the complement to a compact set (only that $H$ is constant outside a compact set). Strangely, this fact is not mentioned in [HZ2] although all the ingredients needed for the proof are already there.

The properties of the relative capacity are summarized in the following

**Theorem 2.1.10.**

1. [Invariance]. *The relative capacity* $\mathrm{c}_{\mathrm{HZ}}$ *is an invariant of symplectomorphisms.*

2. [Monotonicity]. *Let* $Z' \subset Z \subset V \subset V'$. *Then* $\mathrm{c}_{\mathrm{HZ}}(V, Z) \leq \mathrm{c}_{\mathrm{HZ}}(V', Z')$. *In particular,* $\mathrm{c}_{\mathrm{HZ}}(V, Z) \leq \mathrm{c}_{\mathrm{HZ}}(V)$.

3. [Homogeneity]. $\mathrm{c}_{\mathrm{HZ}}(V, Z, a\omega) = a\,\mathrm{c}_{\mathrm{HZ}}(V, Z, \omega)$, *for any constant* $a > 0$.

4. [Normalization]. *Assume that* $M$ *is a closed symplectic submanifold of a geometrically bounded symplectically aspherical manifold* $W$ *and let* $U_r$ *be a symplectic tubular neighborhood of* $M$ *in* $W$ *of radius* $r > 0$ *(see Section 3.1.1 for the definition). Then* $\mathrm{c}_{\mathrm{HZ}}(U_r, M) = \pi r^2$.

Here all assertions but the last one are obvious. Regarding the normalization assertion, we note that Theorem 2.1.3 and Remark 2.1.4 give the upper bound $\mathrm{c}_{\mathrm{HZ}}(U_r, M) \leq \pi r^2$. On the other hand, it is straighforward to construct, for any positive $m < \pi r^2$, a function $H \in \mathcal{H}(U_r, M)$ with $\max H = m$ having no fast periodic orbits. This shows that $\mathrm{c}_{\mathrm{HZ}}(U_r, M) \geq \pi r^2$ and, thus, proves the last assertion.

A number of results on the Weinstein conjecture can be interpreted as calculations of the relative Hofer–Zehnder capacity.

*Example* 2.1.11. Let $(M, \omega)$ be a compact symplectic manifold, and $W = M \times \mathbb{C}^n$. Let $U_r = M \times B_r^{2n}$, where $B_r^{2n}$ is the ball of radius $r > 0$ in $\mathbb{C}^n$. Then, as has been established in [HV2],

$$\mathrm{c}_{\mathrm{HZ}}(U_r, M) = \mathrm{c}_{\mathrm{HZ}}(M \times B_r^{2n}, M \times \{0\}) = \pi r^2,$$

provided that $r > 0$ is small enough. Namely, as long as the inequality

$$\pi r^2 < m(M, \omega, J) = \inf\{\int_{S^2} u^*\omega > 0 \mid u \text{ is a non-constant } J\text{-holomorphic sphere}\}$$

holds. Note that the invariant $m(M, \omega, J)$ is always positive by Gromov's compactness theorem. If $\omega|_{\pi_2(M)} = 0$, this holds for any $r > 0$. Note also that when $W$ is symplectically aspherical, this can be proved similarly to the last assertion of Theorem 2.1.10. Furthermore, when $\omega|_{\pi_2(M)} = 0$, we also have $c_{\mathrm{HZ}}(U_r) = \pi r^2$ as is proved in [FHV, Ma]. Hence, in this case, $c_{\mathrm{HZ}}(U_r, M) = \pi r^2 = c_{\mathrm{HZ}}(U_r)$. We also refer the reader to [LT, Lu1] for further results in this direction.

*Example* 2.1.12. Let $(M, \omega)$ be a closed symplectic manifold. Then, as an immediate consequence of a theorem in [HV2], we have

$$c_{\mathrm{HZ}}(M \times S^2, M \times \{0\}) = \mathrm{Area}(S^2),$$

provided that

$$\mathrm{Area}(S^2) < m(M, \omega) = \inf\{\int_{S^2} u^*\omega > 0 \mid u\colon S^2 \to M\}.$$

In contrast with $m(M, \omega, J)$ from the previous example, here $m(M, \omega)$ can be zero. For instance, this is the case for $S^2 \times S^2$ with two unequal symplectic forms.

*Example* 2.1.13. In general, the relative capacity $c_{\mathrm{HZ}}(V, Z)$ need not be equal to the capacity $c_{\mathrm{HZ}}(V)$. For example, $c_{\mathrm{HZ}}(V, Z) < c_{\mathrm{HZ}}(V)$ whenever $c_{\mathrm{HZ}}(V) < \infty$ and $Z$ is the closure of an open subset of $V$; see Section 3.3.2 and, in particular, Proposition 3.3.4. Furthermore, it appears that the two capacities may differ for small neighborhoods of a Lagrangian submanifold $Z$.

**Almost existence**

As for the ordinary Hofer–Zehnder capacity, the finiteness of relative capacity implies almost existence of periodic orbits:

**Theorem 2.1.14 (Relative almost existence theorem).** *Assume that* $\mathrm{c}_{\mathrm{HZ}}(V, Z) < \infty$ *and let* $H: V \to \mathbb{R}$ *be a proper smooth function constant on* $Z$ *and such that* $H|_Z = \min H$. *Then for almost all (in the sense of measure theory) regular values* $c$ *in the range of* $H$, *the level* $H = c$ *carries a periodic orbit.*

The proof of this theorem, which is omitted here, is identical to the proof of the standard almost existence theorem, see [HZ2, Section 4.2]. We will further discuss the almost existence theorem and some elements of its proof in Section 3.3.2.

The notion of capacity considered above does not allow one to control the homotopy class of periodic orbits. There are a number of ways to deal with this shortcoming; see [BPS, Sc]. Here we take an alternative approach by restricting our attention to a collection of subsets of a fixed symplectic manifold $V$. The resulting notion which henceforth we refer to as the restricted relative Hofer–Zehnder capacity, even though lacking the flexibility of the capacities introduced in [BPS, Sc], is very simple and sufficient for our purpose to detect periodic orbits contractible in the ambient manifold $V$.

**Restricted Hofer–Zehnder capacity**

Let, as above, $(V, \omega)$ be a symplectic manifold, $Z$ a compact subset of $V$, and $U$ an open subset of $V$ containing $Z$.

**Definition 2.1.15.** The *restricted relative Hofer–Zehnder capacity*

$$\bar{\mathrm{c}}_{\mathrm{HZ}}(U, Z) \in (0, \infty]$$

is defined as $\sup \max H$, where the supremum is taken over $H \in \mathcal{H}(U, Z)$ such that all non-trivial, *contractible in* $V$, periodic orbits of $H$ are slow. When the dependence of the restricted capacity on $\omega$ is essential, we will use the notation $\bar{\mathrm{c}}_{\mathrm{HZ}}(U, Z, \omega)$.

By definition, the restricted capacity is an invariant of $(U, Z)$ with respect to symplectomorphisms of the ambient manifold $V$. Furthermore, it is clear that

$$\mathrm{c}_{\mathrm{HZ}}(U, Z) \leq \bar{\mathrm{c}}_{\mathrm{HZ}}(U, Z)$$

and

$$\mathrm{c}_{\mathrm{HZ}}(U, Z) = \bar{\mathrm{c}}_{\mathrm{HZ}}(U, Z) \quad \text{if } U \text{ is simply connected.}$$

Assume now that the ambient manifold is geometrically bounded and symplectically aspherical. In accordance with our conventions, we denote it by $W$.

**Theorem 2.1.16.** *Theorems 2.1.8, 2.1.10, and 2.1.14 hold, with obvious modifications (e.g., $V$ replaced by $U$) for the restricted relative Hofer–Zehnder capacity. In particular,*

$$\mathrm{c}_{\mathrm{HZ}}(U_r, M) = \pi r^2 = \bar{\mathrm{c}}_{\mathrm{HZ}}(U_r, M), \tag{2.1.2}$$

*when $M$ is a closed symplectic submanifold of a geometrically bounded symplectically aspherical manifold $W$. In the almost existence theorem, the flow of $H$ has contractible in $W$ non-trivial periodic orbits on the level $H = c$ for almost all regular values $c$ in the range of $H$.*

This result can be easily verified by scrutinizing the proofs of Theorems 2.1.8, 2.1.10, and 2.1.14. The normalization assertion, i.e., (4.4.9), readily follows from the fact that Theorem 2.1.3 guarantees the existence of contractible periodic orbits.

As a consequence, we have

*Corollary* 2.1.17. Let $M$ be a closed symplectic submanifold of $W$ and let $H\colon W \to \mathbb{R}$ attain its absolute minimum $H = 0$ on $M$. Then the levels $H = \epsilon$ carry contractible in $W$ periodic orbits for almost all small $\epsilon > 0$.

This corollary generalizes a theorem of [CGK] where the existence of periodic orbits is established for a dense set of small $\epsilon > 0$. Note also that another version of an almost existence theorem for convex manifolds (also using the notion of relative capacity) has been recently proved in [FS1].

*Remark* 2.1.18. As in [HZ2], it is easy to see that the constant in Theorem 2.1.1 is in fact equal to $\bar{c}_{\mathrm{HZ}}(U, M)$ and there exists a contractible periodic orbit of period no greater than $\min_M H/\bar{c}_{\mathrm{HZ}}(U, M)$. (When the orbits are not required to be contractible, one should replace $\bar{c}_{\mathrm{HZ}}(U, M)$ by $c_{\mathrm{HZ}}(U, M)$.)

### 2.1.3 Application: the motion of a charge in a magnetic field

Let $M$ be a closed Riemannian manifold and let $\sigma$ be a closed two-form (magnetic field) on $M$. Equip $W = T^*M$ with the twisted symplectic structure $\omega = \omega_0 + \pi^*\sigma$, where $\omega_0$ is the standard symplectic form on $T^*M$ and $\pi\colon T^*M \to M$ is the natural projection. It is known that $(W, \omega)$ is geometrically bounded for any $\sigma$. (We refer the reader to [AL, CGK, Lu1] for a discussion of this question.) Finally, let $H$ be the standard kinetic energy Hamiltonian on $T^*M$. The Hamiltonian flow of $H$ on $W$, called a *twisted geodesic flow*, is of interest because it describes, for example, the motion of a charge on $M$ in the magnetic field $\sigma$.

In this setting, as a particular case of Corollary 2.1.17, we have

*Corollary* 2.1.19. Assume that $(M, \sigma)$ is symplectically aspherical. Then for almost all small $\epsilon > 0$, the energy level $H = \epsilon$ carries a periodic orbit whose projection to $M$ is contractible.

*Example* 2.1.20. Let $M$ be a closed surface with a metric of constant curvature equal to $-1$ and let $\sigma$ be the area form on $M$. All orbits of the twisted geodesic flow on the levels $H = \epsilon$ are periodic and contractible in $T^*M$ for $0 < \epsilon < 1/2$; for $\epsilon = 1/2$, the level carries no periodic orbits; and for $\epsilon > 1/2$ the flow is smoothly equivalent to the geodesic flow on the unit cotangent bundle $ST^*M$. In particular, the levels $H = \epsilon$ with $\epsilon > 1/2$ carry no periodic orbit with contractible projections to $M$. (We refer the reader to, e.g., [Gi2] for proofs of these standard facts and further references.) As a consequence, we see that Corollary 2.1.19 does not extend to large values of $\epsilon$ as long as the orbits are required to have contractible projections. Likewise, Theorem 2.1.3 for contractible orbits does not extend to arbitrarily large bounded neighborhoods of $M$. Furthermore, set $W_\epsilon = \{H < \epsilon\}$. Now, it is easy to see from Theorem 2.1.3 that $c_{\mathrm{HZ}}(W_\epsilon, M) = \bar{c}_{\mathrm{HZ}}(W_\epsilon, M) < \infty$ as long as $0 < \epsilon < 1/2$. For $\epsilon > 1/2$, we have $\bar{c}_{\mathrm{HZ}}(W_\epsilon, M) = \infty$ and it is not known if $c_{\mathrm{HZ}}(W_\epsilon, M)$ is still finite.

Corollary 2.1.19 strengthens or complements a number of other results on periodic orbits of twisted geodesic flows. Under the hypotheses of the corollary, the existence of a dense set of low energy levels with contractible periodic orbits has been proved in [CGK]. It is also known that when $\sigma$ is symplectic, but under no assumptions on $\pi_2(M)$, there exists a sequence of energy values $\epsilon_k \to 0$ such that the levels $H = \epsilon_k$ carry contractible periodic orbits, [GK2]. The same is true for any $\sigma \neq 0$, provided that $\pi_2(M) = 0$, [Mac1, Pol2]. When $M$ is a torus, periodic orbits (not necessarily contractible, e.g., if $\sigma = 0$) exist on almost all energy levels for any $\sigma$. In fact, in this case the ordinary Hofer–Zehnder capacity of any bounded domain is finite; see, e.g., [GK1] and references therein, cf. [Ji]. This is also true for the restricted Hofer–Zehnder capacity of small neighborhoods of $M$, when $\sigma \neq 0$ is exact and $\pi_2(M) = 0$, [FS1].

Furthermore, under suitable additional conditions, every low energy level admits a number of periodic orbits when $\sigma$ is symplectic. We refer the reader to [GK1, Ke1] for some recent results and to, e.g., the survey [Gi2] and references therein for a discussion of the results obtained prior to 1995.

Corollary 2.1.19 and the results quoted above suggest that, for any closed $\sigma$, every low energy level $H = \epsilon$ carries a periodic orbit. However, this conjecture cannot be established by purely symplectic topology methods: one should make use, in an essential way, of the fact that $H$ is convex along the fibers, as the results of Subsection 2.3.2 indicate.

## 2.2 Time-dependent Hamiltonian flows

So far we have been concentrating on the existence problem for non-trivial periodic orbits of autonomous Hamiltonians. The main technical tool in our dealing with this problem has been Theorem 2.1.3 which guarantees that an independent of time Hamiltonian $H \in \mathcal{H}(W, M)$ with sufficiently large variation has a non-trivial contractible one-periodic orbit.

An analogue of this result holds for time-periodic Hamiltonians. In this setting, which is similar to that of Arnold's conjecture, the requirement that the orbit be non-trivial appears to make little sense. Furthermore, once this requirement is dropped, the existence result can be established under conditions much less restrictive than in Theorem 2.1.3. Below we prove only the simplest version of the theorem one can expect to hold for time-dependent Hamiltonians (see Remark 3.2.3), for our goal is to emphasize the difference between the autonomous and time-dependent cases rather than to analyze the time-dependent case in detail.

As in Section 2.1, let $W$ be a geometrically bounded symplectically aspherical manifold and let $M$ be a closed symplectic submanifold of $W$. Consider a one-periodic in time Hamiltonian $H \colon S^1 \times W \to \mathbb{R}$ supported in $S^1 \times U$, where $U$ is an open set containing $M$. Then, along the lines of [BPS], we have the following analogue of Theorem 2.1.3:

**Theorem 2.2.1.** *Assume that $U$ is a sufficiently small neighborhood of $M$. Then for every $H \geq 0$ supported in $S^1 \times U$ and such that $\min_{S^1 \times M} H > 0$, the Hamiltonian flow of $H$ has a contractible one-periodic orbit with positive action. If the periodic orbits of $H$ with positive action are non-degenerate, the number of these orbits is no less than the sum of Betti numbers of $M$ with $\mathbb{Z}_2$ coefficients.*

This theorem is an immediate consequence of the following

**Theorem 2.2.2.** *Assume that $U$ is a sufficiently small neighborhood of $M$ and let $H \geq 0$, supported in $S^1 \times U$, be such that $\min_{S^1 \times M} H > 0$. Then, for every small $a > 0$, there exists* *an epimorphism* $\mathrm{HF}^{[a,\infty)}(H) \to H_*(M;\mathbb{Z}_2)$.

Theorem 2.2.2 will be proved in Section 3.2.

Theorem 2.2.1 holds under much weaker hypotheses than its autonomous counterpart, Theorem 2.1.3. However, the trade-off is that Theorem 2.2.1 does not guarantee the existence of non-trivial periodic orbits. As is well known, for any $C^2$-small autonomous Hamiltonian $H$, one-periodic orbits of the flow of $H$ are trivial (cf. the construction in Example 3.2.4 with $c > 0$ small). Moreover, in sharp contrast with Theorem 2.1.3, even when $M$ is a point in the setting of Theorem 2.2.1, there exists a Hamiltonian $H$ with arbitrarily large $\min_{S^1\times M} H$ such that all of its positive actions are arbitrarily small (see Example 3.2.5).[1] In other words, the value of $a$ in Theorem 2.2.2 cannot be expressed in terms of $\min_{S^1\times M} H$ only. Furthermore, Example 3.2.5 also shows that there exists a compactly supported Hamiltonian $H$ (not satisfying $H \geq 0$) such that $\min_{S^1\times M} H$ is arbitrarily large but all the action values for $H$ are non-positive and the time-one flow of $H$ is non-trivial. These examples demonstrate that for a small neighborhood of $M$ the relative symplectic capacities introduced in [BPS] are trivial.

Further comparing Theorems 2.2.1 and 2.2.2 with the results of [BPS] note that when $M$ is a Lagrangian submanifold and the orbits are sought in a non-trivial homotopy class $\alpha$, the assumption $\min_{S^1\times M} H > C_\alpha > 0$ from [BPS] is clearly logical and necessary. However, when the orbits looked for are contractible the role of the condition $\min_{S^1\times M} H > 0$ is less apparent, in particular, if the requirement that the action is positive replaced by that the action is non-zero.

More specifically, it is possible that the first assertion of Theorem 2.2.1 still holds when $H \geq 0$, but the condition that $\min_{S^1\times M} H > 0$ is replaced by the requirement that $H$ is not identically zero. When $W$ is convex, this can be proved using the monotonicity of the Schwarz action selector (cf. [FS1, Sc]). When $H$ is not assumed to be non-negative, the

[1] The author is grateful to Leonid Polterovich for this remark.

assumption that the time-one flow of $H$ is non-trivial should be added and in this case one should look for the orbits with non-zero action. This would lead to a partial generalization of theorem of C. Viterbo, [Vi2]. (Note that when $H \geq 0$ is not identically zero, the time-one flow is non-trivial, at least when the symplectic manifold is exact, since the Calabi invariant of the flow is positive; see [McDS].) For convex manifolds, this version of Viterbo's theorem has been proved in [FS1]. However, the methods of [FS1] heavily rely on a somewhat different definition of Floer homology, making use of convexity and resulting in the homology defined for all intervals of actions. This approach does not readily extend to geometrically bounded symplectic manifolds.

## 2.3 Counterexamples: Hamiltonian flows without periodic orbits

In this section we turn to examples of Hamiltonian flows without periodic orbits. Theorem 2.3.1 of the first subsection shows that a regular level set of a proper Hamiltonian does not necessarily carry a periodic orbit of the Hamiltonian flow, even though periodic orbits exist in abundance on nearby levels. Our results in Subsection 2.3.2 indicate that one cannot expect to prove the existence of periodic orbits on all low energy levels (in the setting of Corollary 2.1.19) by utilizing strictly methods from symplectic topology.

### 2.3.1 Hamiltonian Seifert conjecture in dimension four

Recall that *characteristics* on a hypersurface $M$ in a symplectic manifold $(W, \eta)$ are, by definition, the (unparameterized) integral curves of the field of directions $\ker(\eta|_M)$.

Let $\mathbb{R}^{2n}$ be equipped with its standard symplectic structure.

**Theorem 2.3.1.** *There exists a $C^2$-smooth embedding $S^3 \hookrightarrow \mathbb{R}^4$ which has no closed characteristics. This embedding can be chosen $C^0$-close and $C^2$-isotopic to an ellipsoid.*

As an immediate consequence we obtain

**Theorem 2.3.2.** *There exists a proper $C^2$-function $F\colon \mathbb{R}^4 \to \mathbb{R}$ such that the level $\{F = 1\}$ is regular and the Hamiltonian flow of $F$ has no periodic orbits on $\{F = 1\}$. In addition, $F$ can be chosen so that this level is $C^0$-close and $C^2$-isotopic to an ellipsoid.*

*Remark* 2.3.3. Regular levels of $F$ without periodic orbits are exceptional in the sense that the set of corresponding values of $F$ has zero measure. This is a consequence of the almost existence theorem, [HZ1, HZ2, St], which guarantees that for a $C^2$-smooth (and probably even $C^1$-smooth) function, periodic orbits exist on a full measure subset of the set of regular values. In particular, since all values of $F$ near $F = 1$ are regular, almost all levels of $F$ near this level carry periodic orbits.

*Remark* 2.3.4. It is quite likely that our construction gives an embedding $S^3 \hookrightarrow \mathbb{R}^4$ without closed characteristics, which is $C^{2+\alpha}$*-smooth.*

*Remark* 2.3.5. Similarly to its higher-dimensional counterparts, [Gi1, Gi3], Theorem 2.3.1 extends to other symplectic manifolds as follows. Let $(W, \eta)$ be a four-dimensional symplectic manifold and let $i\colon M \hookrightarrow W$ be a $C^\infty$-smooth embedding such that $i^*\eta$ has only a finite number of closed characteristics. Then there exists a $C^2$-smooth embedding $i'\colon M \hookrightarrow W$, which is $C^0$-close and isotopic to $i$, such that ${i'}^*\eta$ has no closed characteristics.

The idea of the proof, given in Chapter 4, is to adjust Schweitzer's construction, [Schwe], of an aperiodic $C^1$-flow on $S^3$ to make it embeddable into $\mathbb{R}^4$ as a Hamiltonian flow. This is done by introducing a Hamiltonian version of Schweitzer's plug. More specifically, the flow on Schweitzer's plug is defined as the Hamiltonian flow of a certain multi-valued function $K$ which we use to find a symplectic embedding of the plug (see Proposition 4.1.2 and Remark

4.1.4). The existence of such a function $K$ heavily depends on the choice of a Denjoy vector field in Schweitzer's plug. Namely, the Denjoy vector field is required to be essentially as smooth as a Denjoy vector field can be (see Remark 4.4.2). Implicitly, the idea to define the flow on Schweitzer's plug using the Hamilton equation goes back to G. Kuperberg's paper [KuG].

As of this moment we do not know if G. Kuperberg's flow can be embedded into $\mathbb{R}^4$. The two constructions differ in an essential way. The Denjoy flow and the function $K$ in G. Kuperberg's example are required to have properties very different from the ones we need. As a consequence, our method to embed the plug into $\mathbb{R}^4$ does not apply to G. Kuperberg's plug. (For example, one technical but essential discrepancy between the methods is as follows. In G. Kuperberg's construction, it is important to take a rotation number which cannot be too rapidly approximated by rationals, while the Denjoy map is not required to be smoother than just $C^1$. On the other hand, in our construction the value of rotation number is irrelevant, but the smoothness of the Denjoy map plays a crucial role.)

### 2.3.2 Flows on twisted cotangent bundles

We start with an example showing that in Corollary 2.1.17, already for $W = \mathbb{R}^{2n}$, one cannot expect to have periodic orbits on all levels of $H$ unless the minimum of $H$ is non-degenerate. Note that in the non-degenerate case every low energy level $H = \epsilon$ carries at least $n$ distinct periodic orbits. This is the Weinstein–Moser theorem; see [Mo, We].

**Proposition 2.3.6.** *For $2n \geq 6$, there exists a proper $C^\infty$-function $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ whose only critical point is the origin, where $H$ has an absolute minimum (say, $H(0) = 0$), and such that the levels $H = \epsilon_k$ carry no periodic orbits for some sequence of regular values $\epsilon_k \to 0$. The levels $H = \epsilon_k$ are isotopic to the standard sphere in $\mathbb{R}^{2n}$. When $2n = 4$, there is a $C^2$-smooth function with these properties.*

The construction of $H$ as well as the constructions of the other two counterexamples below will be given in Section 4.5.

*Remark* 2.3.7. The function $H$ has a very degenerate (flat) minimum at the origin. One can also modify the construction of $H$ so that $H$ has two critical sets: a non-degenerate minimum $H(0) = 0$ and a critical set $H = \epsilon > 0$, diffeomorphic to $S^{2n-1}$, and $\epsilon_k \to \epsilon$. These results were stated in [Gi5] without proof.

The next counterexample concerns Hamiltonian flows on $T^*S^n$ with the standard symplectic structure.

**Proposition 2.3.8.** *For $n \geq 3$, there exists a $C^\infty$-function $H\colon T^*S^n \to \mathbb{R}$ attaining its absolute minimum $H = 0$ at the zero section and such that for some sequence of regular values $\epsilon_k \to 0$ the levels $H = \epsilon_k$ carry no periodic orbits of the Hamiltonian flow of $H$ with respect to the standard symplectic structure $\omega_0$ on $T^*S^n$. The zero section is the only critical set of $H$ and the levels $H = \epsilon_k$ are isotopic to the unit cotangent bundle $ST^*S^n$. When $n = 2$, there exists a $C^2$-smooth function with these properties. Moreover, the same is true for some exact non-zero magnetic field $\sigma$ on $S^n$, $n \geq 2$.*

This proposition shows that in the results of [HV1, Vi3] concerning the Weinstein conjecture in the cotangent bundles the contact type condition cannot, in general, be omitted.

Finally, let us turn to magnetic flows for non-degenerate magnetic fields. Let $\sigma$ be the standard symplectic form on $M = \mathbb{CP}^n$ and, as above, $\omega = \omega_0 + \pi^*\sigma$ the twisted symplectic form on $W = T^*\mathbb{CP}^n$.

**Proposition 2.3.9.** *The assertions of Proposition 2.3.8 hold when $(T^*S^n, \omega_0)$ is replaced by $(T^*\mathbb{CP}^n, \omega)$ with a $C^\infty$-smooth Hamiltonian $H$ for $n \geq 2$ and a $C^2$-smooth Hamiltonian for $n = 1$.*

*Remark* 2.3.10. In this example, $M$ is not symplectically aspherical. It appears plausible that similar examples exist for symplectically aspherical manifolds.

*Remark* 2.3.11. As in Proposition 2.3.6, the function $H$ in Propositions 2.3.8 and 2.3.9 has a very degenerate (flat) minimum along the zero section.

# Chapter 3

# Proofs: relative almost existence of periodic orbits

## 3.1 Preliminaries

The main goal of Section 3.1 is to recall some background results and definitions needed for the proofs, in particular, those concerning filtered Floer homology.

### 3.1.1 Symplectic tubular neighborhoods

Let $M$ be a compact symplectic submanifold of a symplectic manifold $(V, \omega)$. Denote by $E \to M$ the symplectic normal bundle to $M$. Recall that a neighborhood of the zero section in $E$ has a natural symplectic structure $\omega_E$. Moreover, on this neighborhood, there exists a fiberwise quadratic Hamiltonian whose flow is periodic. This can be seen as follows.

Let us equip $E$ with a Hermitian metric compatible with the fiberwise symplectic structure on $E$ and denote by $\rho\colon E \to \mathbb{R}$ the square of the fiberwise norm, i.e., $\rho(z) = \|z\|^2$.

Recall that $E$ has a canonical fiberwise one-form whose differential is the fiberwise symplectic form. (The value of this form at $z \in E$ is equal to the contraction of the fiberwise symplectic form with $z$.) Fixing a Hermitian connection on $E$, we extend this fiberwise one-form to a genuine one-form $\theta$ on $E$.

Then the form

$$\omega_E = \frac{1}{2} d\theta + \sigma$$

is symplectic on a neighborhood of the zero section in $E$. Here we have identified $\sigma = \omega|_M$ with its pull-back to $E$.

By the symplectic neighborhood theorem, the neighborhood $\{\rho < r^2\}$ of $M$ in $(E, \omega_E)$ is symplectomorphic to a neighborhood of $M$ in $(V, \omega)$ for some $r > 0$. From now on, we denote this neighborhood by $U_r$, assume the identification of $\{\rho < r^2\}$ and $U_r$, and refer to $U_r$ together with this identification as a *symplectic tubular neighborhood of $M$ in $V$ of radius $r$*. In particular, in what follows, $\omega = \omega_E$ and $\rho$ is regarded as a function on a neighborhood of $M$ in $V$. Sometimes, we will write $\rho(x)$ as $\|x\|^2$.

It is not hard to see that all orbits of the Hamiltonian flow of the function $\frac{1}{2}\rho \colon E \to \mathbb{R}$ are periodic with period $2\pi$, just as for the square of the standard norm on $\mathbb{R}^{2n}$. This fact will be essential for the calculation of the Floer homology of a small tubular neighborhood of $M$ in $V$.

### 3.1.2 Floer homology

In this section, we will recall a few facts concerning Floer homology needed for the proofs. The reader interested in a detailed treatment of this material should consult the Floer's papers [Fl1, Fl2, Fl3] or, for example, [HZ2, Sa] for a general introduction to Floer homology, and [BPS, CFH, CFHW, CGK, FH, FHW] for the definition and properties of symplectic homology.

**The definition of filtered Floer homology**

Let, as in Section 2.1, $(W, \omega)$ be a geometrically bounded symplectically aspherical manifold. Denote by $\mathcal{H}$ the space of smooth, compactly supported Hamiltonians $H\colon S^1 \times W \to \mathbb{R}$. To each $H \in \mathcal{H}$, we associate the action functional $A_H$, defined by (2.1.1), on the space of smooth contractible loops in $W$. The critical points $\mathcal{P}(H)$ of $A_H$ are exactly contractible one-periodic orbits of the time-dependent Hamiltonian flow of $H$. The set of critical values of $A_H$ is called the *action spectrum* of $H$ and we denote it by

$$\mathcal{S}(H) = \{A_H(x) \mid x \in \mathcal{P}(H)\}.$$

It is known that $\mathcal{S}(H)$ is compact and nowhere dense in $\mathbb{R}$, [HZ2, Sc].

The Floer homology of $H$ for a certain interval of actions is the homology of the (relative) Morse complex of $A_H$ on the space of all contractible loops. However, when $W$ is not compact, every point in the complement of *supp* $H$ is a degenerate one-periodic orbit, i.e., a critical point of $A_H$, with zero action. To avoid this set, we will only consider the homology generated by the contractible one-periodic orbits with action in an interval that does not contain zero.

To make this description more precise, let us first recall, following [CFH], the definition of the Floer homology $\mathrm{HF}^{[a,\,b)}(H)$ for the negative range of actions $a < b < 0$. For a fixed $a < 0$ such that $a \notin \mathcal{S}(H)$, let

$$\mathcal{P}^a(H) = \{x \in \mathcal{P}(H) \mid A_H(x) < a\}.$$

Assume first that $H$ satisfies the following condition:

$$\text{Every one-periodic orbit } x \in \mathcal{P}^a(H) \text{ is nondegenerate.} \tag{3.1.1}$$

Since $c_1(TM)|_{\pi_2(M)} = 0$, the elements of $\mathcal{P}^a(H)$ are graded by the Conley-Zehnder index $\mu_{\mathrm{CZ}}$

(see, e.g., [Sa]) and the Floer complex of $H$ for actions less than $a$ is the graded $\mathbb{Z}_2$-vector space

$$\mathrm{CF}^a(H) = \bigoplus_{x \in \mathcal{P}^a(H)} \mathbb{Z}_2 x.$$

Note that for the action functional (2.1.1), the Hamiltonian vector field $X_H$ is given by $dH = -i_{X_H}\omega$ (differing by sign from the Hamiltonian vector field in, say, [Sa]) and the direction of $X_H$ effects the sign of Conley–Zehnder indices.

To define the Floer boundary operator, we first fix an almost complex structure $J_{gb}$ for which $(W, \omega)$ is geometrically bounded as in [CGK]. Let $\mathcal{J}$ be the set of smooth $t$-dependent $\omega$-tame almost complex structures which are $\omega$-compatible near $supp(H)$ and are equal to $J_{gb}$ outside some compact set. Each $J \in \mathcal{J}$ gives rise to a positive–definite bilinear form on the space of contractible loops in $W$. We can then consider the moduli space $\mathcal{M}(x, y, H, J)$ of *downward* gradient-like trajectories of $A_H$ which go from $x$ to $y$ and have finite energy. For a dense subset, $\mathcal{J}_{reg}(H) \subset \mathcal{J}$, each moduli space $\mathcal{M}(x, y, H, J)$ is a smooth manifold of dimension $\mu_{\mathrm{CZ}}(x) - \mu_{\mathrm{CZ}}(y)$.

As usual, the Floer boundary operator is then defined by

$$\partial^{H,J} x = \sum_{y \in \mathcal{P}^a(H) \text{ with } \mu_{\mathrm{CZ}}(x) - \mu_{\mathrm{CZ}}(y) = 1} \tau(x, y) y, \tag{3.1.2}$$

where $\tau(x, y)$ stands for the number (mod 2) of elements in $\mathcal{M}(x, y, H, J)/\mathbb{R}$ and $\mathbb{R}$ acts (freely) by translation on the gradient-like trajectories. The operator $\partial^{H,J}$ satisfies $\partial^{H,J} \circ \partial^{H,J} = 0$ and the resulting Floer homology groups $\mathrm{HF}^a(H)$ are independent of the choice of $J \in \mathcal{J}_{reg}(H)$.

*Remark* 3.1.1. Since $(W, \omega)$ with $J_{gb}$ is geometrically bounded and $H$ is compactly supported, there is a uniform $C^0$-bound for the elements of $\mathcal{M}(x, y, H, J)$ (see, for example, Chapter V in [AL]). Hence, the compactness of the appropriate moduli spaces follows by the usual arguments. It is unclear whether or not $\mathrm{HF}^a(H)$ depends on the choice of $J_{gb}$. It is independent of this choice if the set of almost complex structures for which $W$ is geometrically bounded is connected.

For any pair $a < b$, set

$$\mathcal{H}^{a,b} = \{H \in \mathcal{H} \mid a, b \notin \mathcal{S}(H)\}.$$

Assume that $H$ has property (3.1.1) for $b$ (and hence for $a$). (Note that this can only happen when $a < b < 0$.) Then the complexes $\mathrm{CF}^a(H)$ and $\mathrm{CF}^b(H)$ are defined and $\mathrm{CF}^a(H)$ is a subcomplex of $\mathrm{CF}^b(H)$. By definition, $\mathrm{HF}^{[a,\,b)}(H)$ is the homology of the quotient complex $\mathrm{CF}^{[a,\,b)}(H) = \mathrm{CF}^b(H)/\,\mathrm{CF}^a(H)$ with the induced boundary operator.

The set $\mathcal{H}^{a,b}$ is open in $\mathcal{H}$ with respect to the strong Whitney $C^\infty$-topology. Moreover, in each component of $\mathcal{H}^{a,b}$, the functions with the property (3.1.1) holding for $b$ form a dense set. For any function $H \in \mathcal{H}^{a,b}$, we define the Floer homology $\mathrm{HF}^{[a,\,b)}(H)$ as $\mathrm{HF}^{[a,\,b)}(K)$ where $K$ is a small perturbation of $H$ such that $K \in \mathcal{H}^{a,b}$ and $K$ satisfies (3.1.1) for $b$. A version of Floer's continuation then shows that $\mathrm{HF}^{[a,\,b)}(K)$ is independent of $K$ as long as $K$ is close to $H$. Hence, $\mathrm{HF}^{[a,\,b)}(H)$ is well defined.

When $0 < a < b$, the condition (3.1.1) for $a$ or $b$ is never satisfied and we adopt a different, and more naive, approach to the definition of Floer homology. Namely, we simply work with the complex generated by periodic orbits with action in $(a,\,b)$. To be more precise, assume that $H \in \mathcal{H}^{a,b}$ and let

$$\mathcal{P}^{a,b}(H) = \{x \in \mathcal{P}(H) \mid a < A_H(x) < b\}.$$

Also, let us temporarily assume that the following condition holds:

$$\text{Every one-periodic orbit } x \in \mathcal{P}^{a,b}(H) \text{ is nondegenerate.} \tag{3.1.3}$$

Consider the $\mathbb{Z}_2$-vector space

$$\mathrm{CF}^{[a,\,b)}(H) = \bigoplus_{x \in \mathcal{P}^{a,b}(H)} \mathbb{Z}_2 x$$

graded by the Conley–Zehnder index. The differential $\partial^{H,J}$ is defined similarly to (3.1.2), but with summation extending only to $y \in \mathcal{P}^{a,b}$. As before, $\left(\partial^{H,J}\right)^2 = 0$, and we set $\mathrm{HF}^{[a,\,b)}(H)$

to be the cohomology of the resulting complex. Since the functions satisfying (3.1.3) for $a < b$ are dense in $\mathcal{H}^{a,b}$, a small perturbation argument as above allows us to define $\mathrm{HF}^{[a,\,b)}(H)$ for any $H \in \mathcal{H}^{a,b}$.

Note that this construction can also be used when $a < b < 0$ and in this case the two definitions lead to the same complex $\mathrm{CF}^{[a,\,b)}(H)$.

The behavior of Floer homology when the interval of actions is shrunk is described by a long exact sequence. Namely, assume that $a < b < c$, none of these points is in $\mathcal{S}(H)$, and $0 \notin [a,\, c]$. Then we have the exact sequence

$$\ldots \to \mathrm{HF}_*^{[a,\,b)}(H) \to \mathrm{HF}_*^{[a,\,c)}(H) \to \mathrm{HF}_*^{[b,\,c)}(H) \to \mathrm{HF}_{*-1}^{[a,\,b)}(H) \to \ldots.$$

Indeed, for either of the above definitions of the Floer complex, we obviously have the exact sequence of complexes

$$0 \to \mathrm{CF}^{[a,\,b)}(H) \to \mathrm{CF}^{[a,\,c)}(H) \to \mathrm{CF}^{[b,\,c)}(H) \to 0,$$

which induces the required exact sequence in Floer homology.

In particular, this shows that the end points of the interval $[a,\, b]$ can be continuously varied without changing $\mathrm{HF}^{[a,\,b)}(H)$ as long as $a$ and $b$ stay away from $\mathcal{S}(H)$.

*Remark* 3.1.2. We will use the Floer homology $\mathrm{HF}^{[a,\,b)}(H)$ with $0 < a < b$ to prove that the function $H$ from Theorem 2.1.3 has a non-trivial one-periodic orbit with positive action. In this setting, one can easily avoid making use of the above construction by replacing $H$ by $-H$ or by working with cohomology, i.e., considering the differential defined by counting upward gradient-like trajectories. However, we have found the setting of positive functions and downward trajectories visually more pleasing, which has motivated our choice.

**Monotone homotopies and monotone homotopy invariance of Floer homology**

Let $H, K \in \mathcal{H}^{a,b}$ be two functions with $H(t,x) \geq K(t,x)$ for all $(t,x) \in S^1 \times W$. Then there exists a monotone homotopy $s \mapsto K_s$ from $H$ to $K$, i.e., a family of functions $K_s$ such that $s \mapsto K_s(t,x)$ is monotone decreasing for all $(t,x)$ and

$$K_s = \begin{cases} H & \text{for } s \in (-\infty, -1], \\ K & \text{for } s \in [1, \infty). \end{cases}$$

Such a homotopy induces a Floer chain map

$$\mathrm{CF}^{[a,\,b)}(H) \to \mathrm{CF}^{[a,\,b)}(K),$$

and hence a homomorphism of Floer homology

$$\sigma_{KH} \colon \mathrm{HF}^{[a,\,b)}(H) \to \mathrm{HF}^{[a,\,b)}(K).$$

Note that $K_s$ is not required to be in $\mathcal{H}^{a,b}$.

The following facts concerning these homomorphisms are well known; see, e.g., [CFH, FH, Vi3] and [BPS, Sections 4.4 and 4.5]: *The homomorphism $\sigma_{KH}$ is independent of the choice of the monotone homotopy $K_s$ and has the following properties:*

$$\sigma_{KH} \circ \sigma_{HG} = \sigma_{KG} \text{ for } G \geq H \geq K,$$

$$\sigma_{HH} = id \text{ for every } H \in \mathcal{H}^{a,b}.$$

Furthermore, Floer homology is homotopy invariant in the following sense: *Assume that $K_s \in \mathcal{H}^{a,b}$ for all $s \in [0,1]$. Then $\sigma_{KH}$ is an isomorphism.*

This shows that the only way in which the map $\sigma_{KH}$ can fail to be an isomorphism is if periodic orbits, with action equal to $a$ or $b$, are created during the homotopy.

**Calculations of Floer homology**

The main tool used in this paper to calculate Floer homology is a theorem of Poźniak, [Poz], which equates filtered Floer homology and the ordinary homology of a connected Morse–Bott non-degenerate set of periodic orbits.

Recall that a subset $P \subset \mathcal{P}(H)$ is said to be a *Morse–Bott non-degenerate manifold of periodic orbits* if the set $C_0 = \{x(0) \mid x \in P\}$ is a compact submanifold of $W$ and $T_{x_0}C_0 = \ker(D\phi^1_H(x_0) - id)$ for every $x_0 \in C_0$. Here $\phi^1_H$ is the time-one flow of $X_H$.

For such sets of periodic orbits we have the following result which holds for geometrically bounded, symplectically aspherical manifolds.

**Theorem 3.1.3.** (Poźniak, [Poz, Corollary 3.5.4]) *Let $a < b$ be outside of the action spectrum of $H$ and such that $[a, b]$ does not contain zero. Also, suppose that the set $P = \{x \in \mathcal{P}(H) \mid a < \mathcal{A}_H < b\}$ is a connected Morse–Bott manifold of periodic orbits. Then $\mathrm{HF}^{[a,\,b)}(H)$ is isomorphic to $H_*(P; \mathbb{Z}_2)$.*

Note that this isomorphism does not preserve the grading, i.e., $\mathrm{HF}^{[a,\,b)}_*(H) = H_{*-s}(P; \mathbb{Z}_2)$, where the shift $s$ depends on the behavior of $H$ near $P$.

We will also need the following elementary (essentially, trivial) observation which sometimes allows one to extend Poźniak's isomorphism in a particular degree to the case where $P$ is disconnected.

**Lemma 3.1.4.** *Let $a < \gamma < b$ be outside of the action spectrum of $H$ and such that $[a, b]$ does not contain zero.*

1. *Suppose that*

$$\mathrm{HF}^{[\gamma,\,b)}_{n_0+1}(H) = \mathrm{HF}^{[\gamma,\,b)}_{n_0}(H) = 0.$$

*Then the natural map $\mathrm{HF}^{[a,\,\gamma)}_{n_0}(H) \to \mathrm{HF}^{[a,\,b)}_{n_0}(H)$ is an isomorphism.*

*2. Suppose that*

$$\mathrm{HF}^{[a,\,\gamma)}_{n_0-1}(H) = \mathrm{HF}^{[a,\,\gamma)}_{n_0}(H) = 0.$$

*Then the natural map* $\mathrm{HF}^{[a,\,b)}_{n_0}(H) \to \mathrm{HF}^{[\gamma,\,b)}_{n_0}(H)$ *is an isomorphism.*

We will use this lemma in the situation where Poźniak's theorem applies to the intervals $[\gamma,\, b]$ or $[a,\, \gamma]$. Then the Floer homology groups for these intervals vanish in degrees outside of a certain range $[\mu_-,\, \mu_+]$ of Conley–Zehnder indices. Therefore, $\mathrm{HF}^{[a,\,\gamma)}_{n_0}(H) \to \mathrm{HF}^{[a,\,b)}_{n_0}(H)$ is an isomorphism when, for instance, $n_0 > \mu_+$ and $\mathrm{HF}^{[a,\,b)}_{n_0}(H) \to \mathrm{HF}^{[\gamma,\,b)}_{n_0}(H)$ is an isomorphism when $n_0 - 1 > \mu_+$. This line of reasoning has been used in [CGK] to calculate the Floer homology in a given degree for large intervals of actions.

*Proof of Lemma 3.1.4.* To prove the first assertion, consider the exact sequence

$$\mathrm{HF}^{[\gamma,\,b)}_{n_0+1}(H) \to \mathrm{HF}^{[a,\,\gamma)}_{n_0}(H) \to \mathrm{HF}^{[a,\,b)}_{n_0}(H) \to \mathrm{HF}^{[\gamma,\,b)}_{n_0}(K_+).$$

The first and the last group in this sequence vanish. Hence, the middle map in the exact sequence is an isomorphism.

In a similar vein, to prove the second assertion, we consider the exact sequence

$$\mathrm{HF}^{[a,\,\gamma)}_{n_0}(H) \to \mathrm{HF}^{[a,\,b)}_{n_0}(H) \to \mathrm{HF}^{[\gamma,\,b)}_{n_0}(H) \to \mathrm{HF}^{[a,\,\gamma)}_{n_0-1}(H).$$

Here again the first and the last group vanish, and therefore the middle map is an isomorphism.

□

## 3.2 Proofs of Theorems 2.1.3 and 2.2.2

In this section, we prove the main theorems on the existence of one-periodic orbits (both the autonomous and time-dependent cases).

### 3.2.1 Proof of Theorem 2.1.3

First note that it suffices to prove the theorem in the case where $U$ is a tubular symplectic neighborhood, say $U_R$, of $M$ in $W$. Then the theorem and Remark 2.1.4 are consequences of the following result giving the exact value of the constant $C$ for $U_R$.

**Theorem 3.2.1.** *Assume that $M$ is a closed symplectic submanifold of a geometrically bounded symplectically aspherical manifold $W$. Then for $R > 0$ small enough and for every $H \in \mathcal{H}(U_R, M)$ with $\max H > \pi R^2$, the Hamiltonian flow of $H$ has a non-trivial contractible one-periodic orbit with action in the interval $(\max H, \max H + \pi R^2]$.*

This theorem, in turn, is based on

**Proposition 3.2.2.** *Let $H \in \mathcal{H}(U_R, M)$ and $\max H > \pi R^2$. Then there exist functions $K_+$ and $K_-$, supported in $U_R$, such that $K_- < H < K_+$ and*

$$\mathbb{Z}_2 = \mathrm{HF}_{n_0}^{[a,\,b)}(K_+) \to \mathrm{HF}_{n_0}^{[a,\,b)}(K_-) = \mathbb{Z}_2$$

*is an isomorphism for some constants $\max H < a < b$, not in the action spectra of $K_\pm$, and $n_0 = \frac{1}{2}(\operatorname{codim} M - \dim M) + 1$. The functions $K_\pm$ can be chosen so that $b$ is arbitrarily close to $\max H + \pi R^2$.*

To prove Theorem 3.2.1, we factor the monotonicity isomorphism as

$$\mathbb{Z}_2 = \mathrm{HF}_{n_0}^{[a,\,b)}(K_+) \to \mathrm{HF}_{n_0}^{[a,\,b)}(H) \to \mathrm{HF}_{n_0}^{[a,\,b)}(K_-) = \mathbb{Z}_2.$$

It follows that $\mathrm{HF}_{n_0}^{[a,\,b)}(H) \neq 0$. Thus, $H$ must have a contractible one-periodic orbit with action in the interval $[a,\,b)$. Since $a > \max H$, this orbit is non-trivial.

To complete the proof of Theorems 2.1.3 and 3.2.1, it remains to prove the proposition.

### 3.2.2 Proof of Proposition 3.2.2

The idea of the proof is to pick $K_-$ and $K_+$ depending only on $\rho$ and squeezing $H$ from above and below as tightly as possible (see Fig. 1). This will guarantee that $[a, b)$ with the required properties does exist. This interval contains more than one value of the action spectrum of $K_+$ or $K_-$. However, only one one-periodic level of $K_\pm$ contributes $\mathbb{Z}_2$ in degree $n_0$ to the Floer homology of $K_\pm$ and the remaining levels with actions in $[a, b)$ contribute to the Floer homology in degrees either less than $n_0 - 1$ or greater than $n_0 + 1$. Then the exact sequence argument (Lemma 3.1.4) shows that the interval $[a, b)$ can be shrunk to contain only the action value essential in degree $n_0$ without changing the homology in this degree. Applying Poźniak's theorem, we see that $\mathrm{HF}_{n_0}^{[a,\, b)}(K_\pm) = \mathbb{Z}_2$. A similar argument shows that the Floer homology in degree $n_0$ remains constant in the course of a monotone homotopy from $K_+$ to $K_-$ even though $a$ and $b$ do not stay outside of the action spectrum.

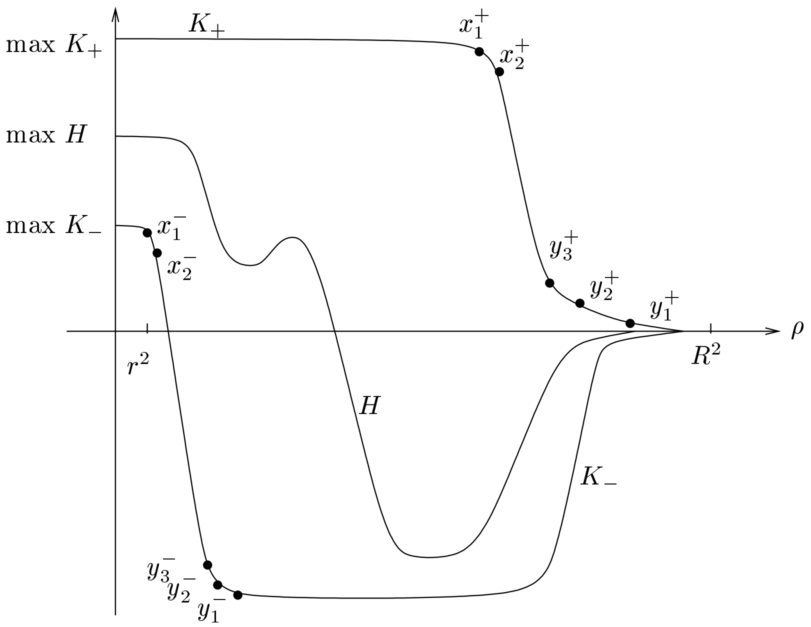
**The definitions of $K_\pm$**

The graphs of functions $K_\pm$ are shown in Fig. 1.[1] These functions depend only on $\rho$ and in what follows we do not distinguish, for the sake of brevity, the functions of $\rho$ from the corresponding functions on $U_R$. The shape of the functions is similar to that used in [BPS], however $\max K_\pm$ are chosen so that these functions bound $H$ from below and above as tightly as possible. Let us now specify some details in the definitions of $K_\pm$.

The function $K_+$ is constant and equal to $\max K_+$ until $\rho$ becomes nearly equal to $R^2$. Then the function rapidly decreases to zero and is identially zero when $\rho$ is very close to $R^2$. The slope of $K_+$, on the interval where this function is non-constant linear, is not an integer multiple of $\pi$. Hence, the Hamiltonian flow of $K_+$ has non-trivial one-periodic orbits

[1] Here we break a recent but well-established tradition to define the functions explicitly and in every detail, as is done for example in [BPS, CGK], and revert to describing only the essential features of the functions, cf. [FHW].

**Figure 3.1**: The functions $K_{\pm}$




on a finite sequence of levels where $\rho$ assumes values:

$$x_1^+ < x_2^+ < x_3^+ < \ldots < \ldots y_3^+ < y_2^+ < y_1^+ < R^2.$$

The points $x_l^+$ are located where the value of the function is still close to $\max K_+$ and the points $y_l^+$ are located where the value of the function is close to zero. Note that $x_1^+ \approx R^2$ due to our choice of $K_+$.

The function $K_-$ is constant and equal to $\max K_-$ until $\rho$ becomes nearly equal to $r^2$ for a sufficiently small constant $r$ such that $0 < r < R$, to be specified later. Then the function rapidly decreases to $\min K_-$. The value $\min K_-$ is chosen so that $K_- < H$. Hence, $\min K_-$ is negative if $H$ assumes negative values and we can take $\min K_- = 0$ if $H \geq 0$. In what follows, we describe $K_-$ in the former case. The function $K_-$ remains constant and equal to $\min K_-$ until $\rho$ nearly reaches $R^2$. Then the function rapidly increases to zero and becomes identically zero for $\rho$ very close to $R^2$. On the intervals where $K_-$ is non-constant linear, the slopes are chosen not to be integer multiples of $\pi$.

The Hamiltonian flow of $K_-$ has non-trivial one-periodic orbits on four finite sequences of levels. The first two of them,

$$r^2 < x_1^- < x_2^- < x_3^- < \ldots < \ldots < y_3^- < y_2^- < y_1^-,$$

are located where $K_-$ is decreasing and $K_- \approx \max K_-$ and $K_- \approx \min K_-$, respectively. Note that $y_1^- \approx r^2$ by the construction of $K_-$. The other two sequences of levels are located where $K_-$ is increasing. For these levels the actions are negative and hence the periodic orbits on these levels do not contribute to the Floer homology $\mathrm{HF}^{[a,\,b)}(K_-)$.

Note that the points $x_l^\pm$ and $y_l^\pm$ are labelled so that the slope of $K_\pm$ increases from $x_l^\pm$ to $x_{l+1}^\pm$ and from $y_l^\pm$ to $y_{l+1}^\pm$, and the periodic orbits on the levels $\rho = x_l^\pm$ and $\rho = y_l^\pm$ have multiplicity $l$.

Particular attention should be given to the choice of $\max K_\pm$. To describe how these maximal values are chosen, denote by $A(x_l^\pm)$ and $A(y_l^\pm)$ the action of $K_\pm$ on the periodic orbits occuring on the levels $\rho = x_l^\pm$ and $\rho = y_l^\pm$, respectively. Then we require that

$$\max K_- < \max H < \max K_+ < A(x_1^-) < A(x_1^+). \tag{3.2.1}$$

Let us show that this choice is possible. The value $\max H$ is given and we also know that $H$ is constant near $M$. Then we can chose $K_-$ and $r > 0$ so that $K_- < H$ near $M$ and

$$\max K_- < \max H < A(x_1^-) \approx \max K_- + \pi r^2.$$

Note that this can be done for an arbitrarily small $r > 0$. Finally, pick $\max K_+$ so that

$$\max H < \max K_+ < A(x_1^-) < A(x_1^+) \approx \max K_+ + \pi R^2.$$

This is clearly possible since $r > 0$ is small and $R$ is fixed.

Finally note that the functions $K_\pm$ are strictly convex or concave outside of the intervals where these functions are constant or linear. This ensures that the energy levels

$\rho = x_l^{\pm}$ and $\rho = y_l^{\pm}$ are Morse–Bott non-degenerate. Furthermore, the functions $K_{\pm}$ can be chosen so that all action values $A(x_l^{\pm})$ and $A(y_l^{\pm})$ are distinct.

**Periodic orbits, actions and Conley–Zehnder indices for $K_{\pm}$**

When $[\alpha, \beta)$ is an interval of (positive) actions containing only one of the points $A(x_l^{\pm})$ and $A(y_l^{\pm})$, the Floer homology $\mathrm{HF}^{[\alpha,\beta)}(K_{\pm})$ can be determined by Poźniak's theorem, [Poz], (see Theorem 3.1.3). Namely, we have

$$\mathrm{HF}_*^{[\alpha,\beta)}(K_{\pm}) = H_{*-s}(SM; \mathbb{Z}_2), \tag{3.2.2}$$

where $SM$ is the unit normal sphere bundle to $M$ in $W$. The shift $s$ depends on whether the function is increasing or decreasing and concave or convex near $x_l^{\pm}$ or $y_l^{\pm}$ and on the multiplicity of the orbits on the level. Hence, the Floer homology in (3.2.2) can be non-zero only for the range of degrees bounded by the Conley–Zehnder indices $*$ such that $* - s = 0$ and $* - s = \dim SM$. These degree ranges and approximate values of actions $A(x_l^{\pm})$ and $A(y_l^{\pm})$ are given in Table 1. Here we use the notations

$$2m = \dim M \quad \text{and} \quad 2n = \operatorname{codim} M, \quad \text{so that} \quad n_0 = n - m + 1,$$

and the dots in the expressions for actions stand for the terms which can be made arbitrarily small by a suitable choice of $K_{\pm}$ while keeping $\max K_{\pm}$ and $r$ and $R$ constant; namely by shortening the convexity/concavity intervals. The calculation of the actions and degree ranges from Table 1 is straightforward (but somewhat tedious for the degrees). For the sake of completeness we will outline it in Section 3.2.2 at the end of the proof.

Returning to the definition of $K_{\pm}$, observe that since $\max H > \pi R^2$, the function $K_+$ can be chosen so that

$$A(y_1^+) < \max H < \max K_+ < A(x_1^+) \quad \text{and} \quad A(y_2^+) < A(x_1^+). \tag{3.2.3}$$

**Table 3.1**: The Conley–Zehnder indices and actions for $x_l$ and $y_l$.

| $\rho$ | Degrees | Actions for $K_+$ | Actions for $K_-$ |
|---|---|---|---|
| $x_1$ | $[n-m+1,\, 3n+m]$ | $\max K_+ + \pi R^2 \pm \dots$ | $\max K_- + \pi r^2 \pm \dots$ |
| $x_2$ | $[3n-m+1,\, 5n+m]$ | $\max K_+ + 2\pi R^2 \pm \dots$ | $\max K_- + 2\pi r^2 \pm \dots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $x_l$ | $[(2l-1)n-m+1,$ $(2l+1)n+m]$ | $\max K_+ + l\pi R^2 \pm \dots$ | $\max K_- + l\pi r^2 \pm \dots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $y_1$ | $[n-m,\, 3n+m-1]$ | $\pi R^2 \pm \dots$ | $\min K_- + \pi r^2 \pm \dots$ |
| $y_2$ | $[3n-m,\, 5n+m-1]$ | $2\pi R^2 \pm \dots$ | $\min K_- + 2\pi r^2 \pm \dots$ |
| $y_3$ | $[5n-m,\, 7n+m-1]$ | $3\pi R^2 \pm \dots$ | $\min K_- + 3\pi r^2 \pm \dots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $y_l$ | $[(2l-1)n-m,$ $(2l+1)n+m-1]$ | $l\pi R^2 \pm \dots$ | $\min K_- + l\pi r^2 \pm \dots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

In a similar vein, since $\min K_- \leq 0$ and $r > 0$ can be taken arbitrarily small, $K_-$ can be chosen so that

$$A(y_1^-) < A(y_2^-) < A(y_1^+) < \max H \tag{3.2.4}$$

Now we are in a position to specify the conditions on the action interval end-points $a$ and $b$. Namely, we only require that these points be outside of the action spectra of $K_\pm$ and

$$\max K_+ < a < A(x_1^-) < A(x_1^+) < b. \tag{3.2.5}$$

In particular, $b$ can be taken arbitrarily large or arbitrarily close to $A(x_1^+)$. Thus, the actions $A(x_1^-)$ and $A(x_1^+)$ are necessarily in the interval $(a,\, b)$ and the interval automatically contains neither the points

$$A(y_1^-) < A(y_2^-) < A(y_1^+) < \max H < \max K_+,$$

nor $\max K_-$. The interval may contain $A(y_2^+)$, but then necessarily $a < A(y_2^+) < A(x_1^+)$. In addition, the interval may contain some of the points $A(x_l^\pm)$ with $l \geq 2$ and some of the points $A(y_l^\pm)$ with $l \geq 3$.

**Showing that** $\mathrm{HF}^{[a,\,b)}_{n_0}(K_\pm) = \mathbb{Z}_2$

Let us first calculate the Floer homology for $K_+$. By our choice of $[a,\,b)$, only the periodic orbits on the levels

$$\rho = x_1^+,\, x_2^+,\, \ldots \quad \text{and} \quad \rho = y_2^+,\, y_3^+,\, \ldots$$

can contribute to the homology.

If the interval $[a,\,b)$ contains only the action $A(x_1^+)$, the identity $\mathrm{HF}^{[a,\,b)}_{n_0}(K_+) = \mathbb{Z}_2$ follows immediately from (3.2.2) and the calculation of the degrees in Table 1. (Note that $n_0$ is exactly the left endpoint of the range of Conley–Zehnder indices for $x_1^\pm$.)

Now we argue inductively (as in [CGK]) to show that the interval $[a,\,b)$ can be shrunk to an interval containing only $A(x_1^+)$. For example, let $b'$ be outside of the action spectrum of $K_+$ and such that

$$a < A(x_1^+) < b' < b$$

and the interval $(b',\,b)$ contains only one point $A(x_l^+)$, $l \geq 2$, or $A(y_l^+)$, $l \geq 3$. (Since $A(y_2^+) < A(x_1^+)$, the action $A(y_2^+)$ cannot occur in the interval $(b',\,b)$.) We need to show that

$$\mathrm{HF}^{[a,\,b')}_{n_0}(K_+) \to \mathrm{HF}^{[a,\,b)}_{n_0}(K_+).$$

is an isomorphism. As can be easily seen from the table,

$$\mathrm{HF}^{[b',\,b)}_{*}(K_+) = 0 \quad \text{for} \quad \begin{cases} * < (2l-1)n - m + 1 & \text{if } A(x_l^+) \in (b',\,b),\, l \geq 2, \\ * < (2l-1)n - m & \text{if } A(y_l^+) \in (b',\,b),\, l \geq 3. \end{cases}$$

Since $n_0 = n - m + 1$, we have $\mathrm{HF}^{[b',\,b)}_{n_0+1}(K_+) = \mathrm{HF}^{[b',\,b)}_{n_0}(K_+) = 0$, and the map in question is an isomorphism by Lemma 3.1.4. (Note that the lemma would not apply when $n = 1$ if we had $A(y_2^+) \in (b', b)$.)

Arguing inductively, we can move $b > A(x_1^+)$ to the left as close to $A(x_1^+)$ as we wish without changing the homology in degree $n_0$.

Next observe that there can be some points $A(y_l^+)$, $l \geq 2$, in the interval $(a,\, A(x_1^+))$. We repeat the same argument. Let $a'$ be such that

$$a < a' < A(x_1^+)$$

and the interval $(a,\, a')$ contains only one of the points $A(y_l^+)$, $l \geq 2$. Then again using the table and applying Lemma 3.1.4, we see that

$$\mathrm{HF}_{n_0}^{[a,\, b)}(K_+) \to \mathrm{HF}_{n_0}^{[a',\, b)}(K_+)$$

is an isomorphism. Indeed, now we need $n_0$ to be outside of the range of degrees for $y_l^+$, i.e., $n_0 < (2l-1)n - m$ for $l \geq 2$, which is clearly true with $y_2^+$ being the worst case scenario. (Then $n_0 - 1$ is automatically outside of the range of degrees.) Therefore, we can move $a < A(x_1^+)$ to the right as close to $A(x_1^+)$ as we wish, without changing the homology in degree $n_0$.

Hence, for the above choice of $[a,\, b)$, the homology $\mathrm{HF}_{n_0}^{[a,\, b)}(K_+)$ is the same as when $[a,\, b)$ contains only $A(x_1^+)$ and we conclude that this group is $\mathbb{Z}_2$.

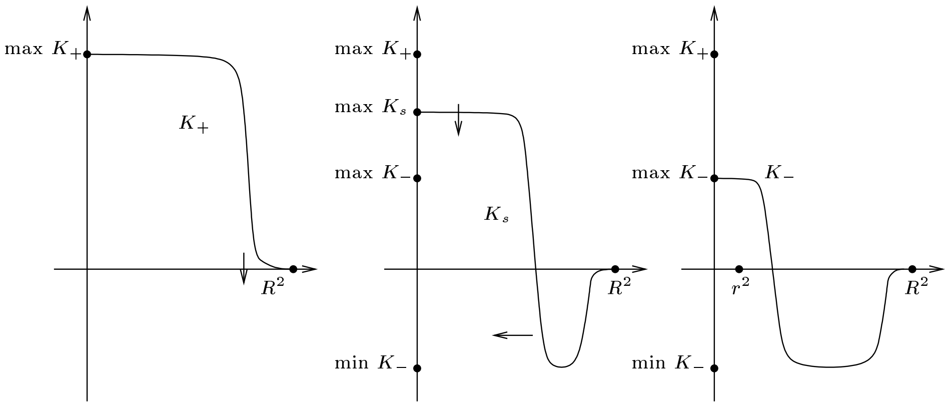
Regarding the calculation of $\mathrm{HF}_{n_0}^{[a,\, b)}(K_-)$, note that the points $y_l^-$ and $y_l^+$ have the same ranges of Conley–Zehnder indices. This is also true for the points $x_l^-$ and $x_l^+$. Furthermore, now not only $A(y_1^-)$ but also $A(y_2^-)$ is outside of the interval $[a,\, b)$. Hence, the argument we have used for $K_+$ translates word-for-word to the calculation of the Floer homology for $K_-$ and, therefore, $\mathrm{HF}_{n_0}^{[a,\, b)}(K_-) = \mathbb{Z}_2$.

**Monotone homotopy and the isomorphism** $\mathrm{HF}_{n_0}^{[a,\, b)}(K_+) \to \mathrm{HF}_{n_0}^{[a,\, b)}(K_-)$

Before describing the monotone homotopy, let us observe that without loss of generality the functions $K_\pm$ can be assumed to have approximately equal slopes and hence equal number of periodic levels $x_l^\pm$ and $y_l^\pm$. This can be achieved by either starting with functions satisfying this requirement or by increasing the slope of one of them through a monotone homotopy. (In the latter case, new periodic levels are created with actions inside of $(a,\, b)$, but the homotopy can be arranged so that $a$ and $b$ stay away from the action spectrum.)

The monotone homotopy $K_s$, $s \in [0,1]$, from $K_0 = K_+$ to $K_1 = K_-$ is shown in Fig. 2. In the course of this homotopy, the bottom part of $K_+$ moves down eventually reaching $\min K_-$. Then, at the second stage of the homotopy, $\max K_+$ moves down to $\max K_-$ in a monotone fashion and the linear part of the function moves to the left. It is easy to see that the homotopy can be arranged so that the periodic energy levels considered above persist under the homotopy and remain Morse–Bott non-degenerate. Thus, $x_l^+$ (or $y_l^+$) moves to $x_l^-$ ($y_l^-$, respectively) through a family of periodic levels $x_l^s$ ($y_l^s$, respectively) and $x_l^s$ and $y_l^s$ are smooth functions of $s \in [0,1]$. Furthermore, as is clear from Fig. 2, the actions $A(x_l^s)$ and $A(y_l^s)$ can be assumed to be monotone decreasing functions of $s$.

**Figure 3.2**: The homotopy from $K_+$ to $K_-$




Our goal is to show that $\mathrm{HF}_{n_0}^{[a,\,b)}(K_s)$ remains constant during the homotopy. If $a$ and $b$ were always away from the action spectrum of $K_s$, this would follow (for all degrees) from the homotopy invariance of Floer homology. However, this is not the case and we have to analyze the behavior of periodic orbits under the homotopy more closely.

For the sake of simplicity, let us first assume that $b$ is to the right of the action spectra of all $K_s$. (Recall that $b$ can be taken arbitrarily large.) Then the actions $A(x_l^s)$, $l \geq 1$, are inside the interval $(a,\, b)$ for all $s \in [0,1]$. Furthermore, $A(y_1^s)$ is outside of $(a,\, b)$ for all $s$. Thus we have

$$A(y_1^s) < a < A(x_1^s) < A(x_2^s) < A(x_3^s) < \ldots < b$$

for all $s$. The actions $A(y_l^s)$ with $l \geq 2$ may cross the left end-point of the interval. Finally, new periodic levels are necessarily created within the interval where $K_s$ is increasing. However, the orbits on these levels have negative actions and hence do not contribute to the homology.

Hence, we only need to examine the effect of $A(y_l^s)$, $l \geq 2$, crossing the left end-point $a$ of the interval at some moment $s_0$. This effect is the same as when moving $a$ through the action value $A(x_l^{s_0})$ in the opposite direction. The exact sequence argument used in the previous section applies in this case, and the homology in degree $n_0$ remains unchanged.

For the sake of completeness, let us outline a rigorous proof of this fact. Let $A(y_l^{s_0}) = a$. Since $A(y_l^s)$ is a monotone decreasing function of $s$, without loss of generality we may assume that there exists a small interval $I = (s_1,\, s_2)$ containing $s_0$ and a small interval $(a_1,\, a_2)$ containing $a$ such that for every $s \in (s_1,\, s_2)$

- $a_1 < A(x_l^{s_1}) < a = A(x_l^{s_0}) < A(x_l^{s_2}) < a_2$ and
- $A(y_l^s)$ is the only point of the action spectrum of $K_s$ in $(a_1,\, a_2)$.

Then the monotone homotopy map

$$\mathrm{HF}_{n_0}^{[a,\, b)}(K_{s_2}) \to \mathrm{HF}_{n_0}^{[a,\, b)}(K_{s_1})$$

factors as

$$\mathrm{HF}_{n_0}^{[a,\, b)}(K_{s_2}) \to \mathrm{HF}_{n_0}^{[a_1,\, b)}(K_{s_2}) \to \mathrm{HF}_{n_0}^{[a_1,\, b)}(K_{s_1}) \to \mathrm{HF}_{n_0}^{[a,\, b)}(K_{s_1}).$$

Here, the first map is an isomorphism because $K_{s_2}$ does not have action spectrum values in $[a_1,\, a]$. The second map is an isomorphism by homotopy invariance. Finally, the third map is an isomorphism by the exact sequence argument. Namely, first note that $y_l^s$ has the same range of indices as $y_l^{\pm}$. Then, using Table 1, we see that $\mathrm{HF}_{n_0-1}^{[a_1,\,a)}(K_{s_1}) = \mathrm{HF}_{n_0}^{[a_1,\,a)}(K_{s_1}) = 0$ and hence, by Lemma 3.1.4, the third map is also an isomorphism.

A similar argument shows that the monotone homotopy induces an isomorphism of the Floer homology in degree $n_0$ for any $b > A(x_1^+)$.

This concludes the proof of Proposition 3.2.2 and of Theorems 3.2.1 and 2.1.3. In the remaining section of the proof, we outline the calculation of the actions and degree ranges given in Table 1.

**Calculation of the actions and degree ranges from Table 1**

The actions $A(x_l^{\pm})$ and $A(y_l^{\pm})$ are easy to determine. For instance, the periodic orbits of $K_+$ on the level $x_l^+$ are the $l$-iterated Hopf circles in the fibers of the symplectic normal bundle to $M$. These circles have radius $\sqrt{x_l^+}$, i.e., approximately $R$, and are traversed in the negative direction, for $K_+$ is decreasing. Thus, the symplectic area bounded by the circles is approximately $-l\pi R^2$ and the value of $K_+$ is near $\max K_+$. Hence, the action is approximately $\max K_+ + l\pi R^2$.

Let us now turn to the ranges of Conley–Zehnder indices. For the sake of brevity we indicate only the main steps of the calculation. Assume first that $M$ is the origin in $W = \mathbb{C}^n$. Considering an explicit perturbation of $K_\pm$ near $x_1^{\pm}$ and $y_1^{\pm}$ and using the definition of the Conley–Zehnder index, it is not hard to see that the range of degrees is $[-n+1,\, n]$ when $K_\pm$ is convex near the level (i.e., for $y_1^{\pm}$) and $[-n,\, n-1]$ when $K_\pm$ is concave near the level (i.e., for $x_1^{\pm}$). These perturbations can be obtained by taking a small non-degenerate quadratic form on $\mathbb{C}P^{n-1}$ and then making a time-dependent perturbation supported near now-isolated periodic orbits. The indices are easy to calculate in the trivializations arising from those of the tangent spaces to $\mathbb{C}P^{n-1}$ at the projections of the orbits. Then, passing to the standard trivialization of $\mathbb{C}^n$ amounts to shifting the range of indices by $2n$. (We refer the reader to, e.g., [Sa] for the definition of the Conley–Zehnder index, its properties, and further references. The reader should keep in mind that the sign convention of [Sa] is different from the one used here and this difference affects Conley–Zehnder indices. For example, for the action defined by (2.1.1), a non-degenerate critical point with Hessian $S$ of an autonomous $C^2$-small Hamiltonian has Conley–Zehnder index $-\text{signature}(S)/2$, but not $\text{signature}(S)/2$ as in [Sa].) For iterated orbits, i.e., the levels $x_l^{\pm}$ and $y_l^{\pm}$, these ranges are further shifted by $2n(l-1)$. This again can be seen as a result of a trivialization change. Finally, to deal with the general case of a tubular neighborhood of $M \subset W$, we perturb $K_{\pm}$ by adding to it a $C^2$-small Morse function on $M$. This results in addition of the interval $[-m, m]$ to the range of Conley–Zehnder indices.

### 3.2.3 Proof of Theorem 2.2.2

As in the proof of Theorem 2.1.3, first observe that it is sufficient to prove the theorem for a tubular neighborhood $U_R$ of $M$. Reasoning as above, we will find functions $K_+$ and $K_-$ (independent of time), supported in $U_R$, such that $K_- < H < K_+$ and

$$\mathrm{HF}^{[a,\,b)}(K_+) \to \mathrm{HF}^{[a,\,b)}(K_-) \tag{3.2.6}$$

is an isomorphism for some constants $0 < a < b$, not in the action spectra of $K_{\pm}$, with $a > 0$ being arbitrarily small. Then the theorem will follow from (3.2.6).

The function $K_+$ has the same shape as its counterpart in Fig. 1 with the only modification that at the top part $K_+$ is now slowly monotone decreasing with constant slope. Thus $K_+$ has a Morse–Bott non-degenerate maximum along $M$ (with small eigenvalues). It is clear that such a function can be chosen so that $H \le K_+$. The function $K_+$ has one-periodic levels at $\rho$ equal to $x_1, \dots$ and $y_1, \dots$ with actions and Conley–Zehnder indices as in Table 1. In addition to this, $M$ is a Morse–Bott non-degenerate set of one-periodic orbits of $K_+$. The range of indices for $M$ is $[n-m, n+m]$ and the action is $\max K_+$. As is clear from Table 1, the action spectrum of $K_+$ is strictly positive except the action on the trivial orbits with $K_+ = 0$.

Let $K_-(\rho) = \epsilon K_+(\epsilon\rho)$, where $0 < \epsilon < 1$. In other words, the graph of $K_-$ is obtained from the graph of $K_+$ by scaling by $\epsilon$. Since $H \geq 0$ and $\min_{S^1 \times M} H > 0$, we clearly have $K_- \leq H$ if $\epsilon > 0$ is small enough.

Consider now the homotopy $K_s(\rho) = sK_+(s\rho)$ with $s \in [\epsilon, 1]$ from $K_+$ to $K_-$. Again, all action values of $K_s$ (on periodic levels where $K_s > 0$) are separated from zero. It follows from homotopy invariance of Floer homology that for any sufficiently large $b$ and any sufficiently small $a > 0$, the monotonicity map

$$\mathrm{HF}^{[a,\,b)}(K_+) \to \mathrm{HF}^{[a,\,b)}(K_-)$$

is an isomorphism in all degrees.

Now it is sufficient to show that $\mathrm{HF}^{[a,\,b)}(K_-) \cong H_*(M; \mathbb{Z}_2)$. To this end, consider the monotone (increasing) homotopy $F_s(\rho) = \epsilon K_+(s\rho)$ with $s \in [\epsilon, r]$. This homotopy begins with $F_\epsilon = K_-$ and ends with $F_r \geq K_-$. (Note that $F_s$ is obtained by dilating the graph of $K_-$ along the $\rho$-axis; the homotopy is defined only if $r > \epsilon$ is small enough, e.g., $r < R$.) In the course of the homotopy $F_s$ from $s = \epsilon$ to $s = r$, the periodic levels $x_l$ and $y_l$ come close to each other, collide, and disappear. No new periodic levels with action close to zero are created. Thus

$$\mathrm{HF}^{[a,\,b)}(F_r) \to \mathrm{HF}^{[a,\,b)}(K_-),$$

is an isomorphism if $b$ is sufficiently large and $a > 0$ is sufficiently small. If $\epsilon > 0$ is small enough and, say, $r = R/2$, all periodic levels $x_l$ and $y_l$ of $F_r$ are destroyed. In other words, $F_r$ does not have one-periodic orbits other than the critical manifold $M$ (in the region where $F_r > 0$). By Poźniak's theorem,

$$\mathrm{HF}^{[a,\,b)}(F_r) \cong H_*(M; \mathbb{Z}_2)$$

with some shift in degrees, which completes the proof of the theorem.

*Remark* 3.2.3. It should be possible to show directly that $\mathrm{HF}^{[a,\,b)}(K_{\pm}) \cong H_*(M;\mathbb{Z}_2)$. Note, however, that since the ranges of Conley–Zehnder indices of $x_1$ and $y_1$ are close to that for $M$, the exact sequence argument utilized in the proof of Proposition 3.2.2 does not apply. Hence, a more delicate reasoning is required, e.g., a calculation of the differential in the Floer complex (cf. [FHW]). Although this approach is more involved than the homotopy argument above, it can perhaps be utilized to relax the hypotheses of Theorem 2.2.1.

### 3.2.4 Discussion

As has been pointed out, Theorem 2.1.3 does not hold if the assumption that $H$ is constant near $M$ is replaced by the weaker assumption that $H$ is constant on $M$. Namely, as the example below shows, for any $R > 0$ and $c > 0$ there exists a smooth non-negative function $H$, constant on $M$ and supported in $U_R$, such that $c = \max H$ and all non-trivial one-periodic orbits of $H$ have action less than or equal to $c$. The Hamiltonian flow of $H$ has numerous non-trivial one-periodic orbits (it should, by Theorem 2.1.1, have a fast non-trivial periodic orbit), but these orbits are not detected by the Floer homology with the range of actions greater than $\max H$.

*Example* 3.2.4. Fix $b = \pi k + \pi/2$, where $k$ is a positive integer, large enough so that $c/b$ (the solution of $c - by = 0$) is in the interval $[0, R^2]$. Let $H = f(\rho)$, where $f$ is a smooth function of the form

$$f(y) = \begin{cases} c - by & \text{for } y \in [0, \delta_-], \\ \text{monotone decreasing} & \text{for } y \in (\delta_-, \delta_+), \\ 0 & \text{for } y \in [\delta_+, \infty). \end{cases}$$

Here, $[\delta_-, \delta_+]$ is an arbitrarily small interval containing $c/b$ and contained in $(0, R^2)$.

Non-trivial one-periodic orbits of $H$ occur on the sphere bundles $\rho = y_l$, where $y_l$ is the solution of the equation

$$f'(y_l) = -\pi l, \quad l \in \mathbb{Z}_+.$$

Since $b$ is not an integer multiple of $\pi$, we have $y_l \in (\delta_-, \delta_+)$ and $l \in [1, b/\pi)$. All orbits on the level $\rho = y_l$ have action

$$\begin{aligned} A_l &= \pi l y_l + f(y_l) \\ &= \pi l y_l + (c - b y_l) + \big(f(y_l) - (c - b y_l)\big) \\ &= c + (\pi l - b) y_l + \big(f(y_l) - (c - b y_l)\big). \end{aligned}$$

It is easy to see that $(\pi l - b) y_l < -\pi\delta_-/2$ and $f(y_l) - (c - b y_l) < c - b\delta_+$ since $y_l \in (\delta_-, \delta_+)$. Thus,

$$A_l - c < -\pi\delta_-/2 + (c - b\delta_+) < 0$$

if $\delta_+$ is taken sufficiently close to $c/b$ while $\delta_-$ is fixed. Therefore, $A_l < c$ for all $l$.

Note also that when $c > 0$ is small, all one-periodic orbits of $H$ are trivial.

*Example* 3.2.5 *(Polterovich).* In this example, due to Leonid Polterovich, we show that in contrast with Theorem 2.1.3, the action value in Theorem 2.2.1 cannot be bounded from below via $\min_{S^1 \times M} H$ even when $M$ is a point. To be more specific, *for any $\epsilon > 0$ and $C > 0$, there exists a non-negative Hamiltonian $H \colon S^1 \times \mathbb{R}^{2n} \to \mathbb{R}$ with arbitrarily small support such that $\min H|_{S^1 \times 0} \geq C$ and every one-periodic orbit of $H$ has action less than $\epsilon$.*

Indeed, first note that the condition that $H$ is periodic in time can be dropped by [BPS, Proposition 2.1.3]. (Here some extra care is needed to keep $H$ non-negative.) Let $K$ be a $C^2$-small non-negative Hamiltonian supported in some ball, such that the time-one flow of $K$ displaces a ball $B$ centered at the origin and such that $\max K < \epsilon$. Let $F$ be a bump-function supported in $B$ with $F(0) \geq C$. Consider the compactly supported Hamiltonian $H_t$ generating the time-dependent flow $\psi_t\varphi_t$, where $\psi_t$ is the flow of $F$ and $\varphi_t$ is the flow of $K$. It is easy to see that $H_t(0) \geq C$ and the only one-periodic orbits of $H_t$ are the critical points of $K$. Hence, all non-zero action values for $H_t$ are less than $\epsilon$.

A similar construction with $K$ now being non-positive shows that *for any* $\epsilon > 0$ *and* $C > 0$*, there exists a Hamiltonian* $H\colon S^1 \times \mathbb{R}^{2n} \to \mathbb{R}$ *with arbitrarily small support such that* $\min H|_{S^1\times 0} \geq C$*, every one-periodic orbit of* $H$ *has a non-positive action, and the time-one flow of* $H$ *is non-trivial.*

*Remark* 3.2.6. We conclude this discussion by pointing out the location of orbits from Theorems 2.1.1 and 2.2.1 for the function $H$ from Example 3.2.4. The orbits from Theorem 2.2.1 lie on the critical manifold $M$. The orbits from Theorem 2.1.1, obtained via Theorem 2.1.3, lie on the level set $\rho = x_1$, in the notations of Table 1. Finally, the orbits which make the homological capacity from [CGK] non-vanish are located on the level $\rho = y_1$ (see also Section 3.3.3). A suitably adapted version of the Schwarz action selector (see [FS1, Sc]) will pick either $\max K_+$ or $A(y_1^+)$, whichever is smaller.

## 3.3 Capacity: proofs and remarks

In Section 3.3, we further discuss the relative Hofer–Zehnder capacity, compare it with some other capacities, and prove its properties stated in Section 2.1.

### 3.3.1 Proof of Theorems 2.1.1 and 2.1.8

The key to the proof of these theorems is the following elementary observation, essentially contained already in [HZ2, p. 184], which allows one to cut off an autonomous Hamiltonian without creating new fast periodic orbits.

**Lemma 3.3.1.** *Let, in the notations of Section 2.1.2,* $H \in \tilde{\mathcal{H}}_0(V, Z)$ *be such that the flow of* $H$

*has no (contractible) non-trivial fast periodic orbits. Then, for any $C$ with $0 < C < \min_Z H$, there exists a function $K \in \mathcal{H}(V,Z)$ such that $C \leq \max K < \min_Z H$ and the flow of $K$ has no non-trivial fast periodic orbits.*

*Remark* 3.3.2. This lemma and Lemma 3.3.3 stated below still hold when *all* fast non-trivial periodic orbits of $H$ and $K$ are replaced by *contractible in $V$* non-trivial fast periodic orbits.

Theorem 2.1.1 immediately follows from Lemma 3.3.1 and Theorem 2.1.3; Remark 2.1.18 is a consequence of Lemma 3.3.1, Theorem 2.1.3, and the definition of relative Hofer–Zehnder capacity.

For the sake of completeness, we outline the proof of Lemma 3.3.1.

*Proof.* Without loss of generality, we may assume that $H \neq 0$. Pick a constant $C < \min_Z H$. Let $\epsilon > 0$ be sufficiently small so that $0 < C - \epsilon$ and $C + \epsilon < \min_Z H$. Then the function $K$ is obtained by cutting $H$ off at the level $H = C$ and then smoothening up the resulting function. More precisely, let $f\colon [C-\epsilon, C+\epsilon] \to \mathbb{R}$ be a function with $0 \leq f' \leq 1$ such that $f(y) = y$ for $y$ near the left end-point of the interval and $f(y) = C$ for $y$ near the right end-point of the interval. Set

$$K(z) = \begin{cases} H(z) & \text{when } H(z) \leq C - \epsilon, \\ f(H(z)) & \text{when } C - \epsilon \leq H(z) \leq C + \epsilon, \\ C & \text{when } C + \epsilon \leq H(z). \end{cases}$$

Then $K$ is constant (and equal to $C$) near $Z$ and equal to $H$ near the boundary of $supp\, H$, i.e., $K \in \mathcal{H}(V,Z)$. Furthermore, $\max K = C$ and $K$ does not have fast periodic orbits since $|f'| \leq 1$. □

Let us prove the first assertion of Theorem 2.1.8, i.e.,

$$c_{\mathrm{HZ}}(V,Z) = \inf E,$$

where

$$E = \{C > 0 \mid \text{every } H \in \tilde{\mathcal{H}}_C(V,Z) \text{ has a non-trivial fast periodic orbit}\}.$$

Note that $E$ is a semi-infinite interval of the form $[\inf E, \infty)$ or $(\inf E, \infty)$ or the empty set. With this in mind, we have $\mathrm{c_{HZ}}(V,Z) \leq \inf E$, by definition. (Indeed, if $H \in \mathcal{H}(V,Z)$ is such that all periodic orbits of $H$ are slow, then $\max H$ must be outside of this interval.) To prove the opposite inequality, assume that $H \in \tilde{\mathcal{H}}_C(V,Z)$ has no fast non-trivial periodic orbits. Then $C$ is in the complement of $E$. By Lemma 3.3.1, for any small $\epsilon > 0$ there exists $K \in \mathcal{H}(V,Z)$ without fast non-trivial periodic orbits and such that $\max K > C - \epsilon$. Thus, $\mathrm{c_{HZ}}(V,Z) \geq C$ and, since the complement of $E$ is also an interval, $\mathrm{c_{HZ}}(V,Z) \geq \inf E$.

Likewise, the second assertion of Theorem 2.1.8 is an easy consequence of

**Lemma 3.3.3.** *Let $H \in \mathcal{H}(V,Z)$ ($H \in \tilde{\mathcal{H}}_C(V,Z)$ with $C > 0$) be such that the flow of $H$ has no non-trivial fast periodic orbits. Then, there exists a non-negative function $K \in \mathcal{H}(V,Z)$ (respectively, $K \in \tilde{\mathcal{H}}_C(V,Z)$) with $\max K = \max H$ having no non-trivial fast periodic orbits.*

*Proof.* Recall that a $C^2$-small function (with support in a given compact set) does not have fast non-trivial periodic orbits; see, e.g., [HZ2, pp. 185, 200]. Using this fact, it is not hard to modify $H$ so that it becomes non-negative near the boundary (in the sense of general topology) of *supp* $H$, without changing $\max H$ or creating fast periodic orbits. We denote the resulting function by $H$ again.

Let $c > 0$ be a small regular value of $H$. Denote by $Y_l, \ldots, Y_k$ the connected components of $\{H \leq c\}$ on which $H$ assumes negative values. Let also $\epsilon > 0$ be so small that $c - \epsilon > 0$ and different connected components $Y_l, \ldots, Y_k$ are contained in different connected components of $\{H \leq c + \epsilon\}$. (As a consequence, $c + \epsilon < \max H$.) Now, separately for each $Y_i$, we cut off $H$ along $\partial Y_i$ and smoothen it up exactly as in the proof of Lemma 3.3.1. It is clear that the resulting function $K$ has the required properties. $\square$

This concludes the proof of the theorems.

### 3.3.2 Almost existence theorem

The proof of the almost existence theorem, both the standard (see [HZ2]) and relative (Theorem 2.1.14) versions, can be broken down into the following two results which may be of independent interest.

**Proposition 3.3.4.** *Let $U$ be an open subset of a symplectic manifold $V$ with compact closure $\bar{U}$ and let $Z \subset U$ be compact. Then*

$$\mathrm{c}_{\mathrm{HZ}}(V, \bar{U}) \leq \mathrm{c}_{\mathrm{HZ}}(V, Z) - \mathrm{c}_{\mathrm{HZ}}(U, Z).$$

*In particular,*

$$\mathrm{c}_{\mathrm{HZ}}(V, \bar{U}) \leq \mathrm{c}_{\mathrm{HZ}}(V) - \mathrm{c}_{\mathrm{HZ}}(U).$$

*Example* 3.3.5. Let $B_R$ be the ball of radius $R > 0$ in $\mathbb{R}^{2n}$, centered at the origin. An argument similar to the proof of Theorem 2.1.3 shows that $\mathrm{c}_{\mathrm{HZ}}(B(R), \bar{B}(r)) = \pi(R^2 - r^2)$. It is not clear whether the inequalities in Proposition 3.3.4 are not in fact equalities.

To further deal with the almost existence problem, let us, following [HZ2], analyze the existence of periodic orbits on a given hypersurface $S$ bounding a domain $U$ in $V$. Let $S_\epsilon$ be a thickening of $S = S_0$ in $V$. Denote by $U_\epsilon$ the domain bounded by $S_\epsilon$. We say that $S$ has *relative Lipschitz type* if

$$\limsup_{\epsilon \searrow 0} \frac{\mathrm{c}_{\mathrm{HZ}}(U_\epsilon, \bar{U})}{\epsilon} < \infty.$$

It is easy to see that this is a well-defined property, i.e., independent of the choice of the thickening $S_\epsilon$. Note that, by Proposition 3.3.4, a Lipschitz type hypersurface (see [HZ2]) is automatically of relative Lipschitz type.

**Proposition 3.3.6.** *A hypersurface of relative Lipschitz type carries a closed characteristic.*

We leave both of these propositions without proofs, for the arguments are implicitly contained in [HZ2, Section 4.2]. Similar results hold for the restricted Hofer–Zehnder capacity $\bar{c}_{HZ}$.

*Remark* 3.3.7. It is not clear if there exist hypersurfaces of relative Lipschitz type which are not of Lipschitz type.

### 3.3.3 Concluding remarks

Arguments used to prove finiteness of a capacity or the existence of periodic orbits can often be turned into a definition of a new capacity, bounding the original one from above; see, e.g., [BPS, CGK, FHW]. The proof of Theorem 2.1.3 is no exception. This leads to the notion of a restricted homological capacity $\bar{c}_{hom}$ which bounds $\bar{c}_{HZ}$ from above and is equal to $\pi r^2$ on $(U_r, M)$. (Here we assume that the ambient manifold $W$ is geometrically bounded and symplectically aspherical; $M$ is a closed symplectic submanifold of $W$.) The definition of this homological capacity is a straightforward (but cumbersome) axiomatization of the proof, and we omit it here. The only advantage the capacity $\bar{c}_{hom}$ seems to have over $\bar{c}_{HZ}$ is that $\bar{c}_{hom}$ gives some information about the actions of periodic orbits. It is not clear, however, how to use this extra information.

Let us briefly discuss the relation of the capacity $c_{HZ}$ or $\bar{c}_{HZ}$ with the capacities introduced in [BPS] and [CGK].

The restricted relative capacity of [CGK] is a relative version of the homological capacity from [FHW]. The finiteness of this capacity results in the "nearby existence theorem" – the existence of periodic orbits on a dense set of levels (see [CGK]) – but falls short of leading to the almost existence theorem. We are not aware of any inequalities relating this capacity with $\bar{c}_{HZ}$ or $\bar{c}_{hom}$.

For the trivial homotopy class, the relative capacity (homological or not) of [BPS] does not allow one to control whether the periodic orbit detected by the capacity is trivial or not (cf. Example 3.2.4). Thus, for the trivial homotopy class this capacity appears to be unrelated to $c_{HZ}$. On the other hand, when the homotopy class is non-zero, the orbit is automatically non-trivial and in this case the capacity of [BPS] gives an upper bound for $c_{HZ}$ (but not $\bar{c}_{HZ}$).

*Example* 3.3.8. Let $Z$ be a closed Lagrangian submanifold of a geometrically bounded symplectically aspherical manifold $W$. When $Z$ is a torus or admits a metric of negative sectional curvature, $c_{HZ}(U, Z) < \infty$, where $U$ is a small neighborhood of $Z$, as immediately follows from the results of [BPS]. The results of [HV1, Vi3] also suggest, but apparently do not imply, that $c_{HZ}(U, Z) < \infty$ in general for a closed Lagrangian submanifold.

Note also that in this case $\bar{c}_{HZ}(U, Z) = \infty$, when $Z$ admits a metric without contractible geodesics and $\pi_1(Z) \to \pi_1(W)$ is a monomorphism. The restricted capacity $\bar{c}_{HZ}$ is specifically "tuned up" to detect contractible periodic orbits near a compact submanifold and Theorem 2.1.3 guarantees the existence of such orbits near a symplectic submanifold. For a general submanifold, such periodic orbits may fail to exist, and the restricted capacity $\bar{c}_{HZ}$ may be infinite even when non-contractible orbits exist in abundance.

As was pointed out by L. Polterovich, one may expect that $c_{HZ}(V, M) = c_{HZ}(V)$ when $M$ is a symplectic submanifold or even when $\omega|_M \neq 0$ and some natural topological conditions hold, e.g., the normal bundle to $M$ in $V$ admits a non-vanishing section; cf. [Pol1] and Example 2.1.11. (Note that finiteness of $c_{HZ}(V)$ or $\bar{c}_{HZ}(V, point)$ leads to a non-relative, stronger than Theorem 2.1.14, almost existence theorem in $V$.) Along these lines, E. Kerman has recently shown, [Ke3], that $\bar{c}_{HZ}(U_r, M) = \bar{c}_{HZ}(U_r)$ for $r > 0$ small, provided that $M$ is a closed rational symplectic submanifold of $W$ and the homology of the unit normal bundle to $M$ splits.

*Remark* 3.3.9 *(Capacity of the cylinder and ellipsoids).* It is not surprising that the method used in the proof of Theorem 2.1.3 readily lends itself for calculations of capacities of some other manifolds. For example, one can easily recover the well-known calculation of the Hofer–Zehnder capacity of ellipsoids (see, e.g., [HZ2]), bypassing the calculation for the cylinder, and then derive from it the result for the cylinder. Namely, consider the solid ellipsoid $U = \{z \in \mathbb{C}^n \mid Q(z) < 1\}$, where

$$Q(z) = \frac{|z_1|^2}{r_1^2} + \cdots + \frac{|z_n|^2}{r_n^2}$$

and $0 < r_1 \leq \ldots \leq r_n$. As is easy to see, $\mathrm{c}_{\mathrm{HZ}}(U) \geq \pi r_1^2$. On the other hand, arguing as in the proof of Proposition 3.2.2 with $K_\pm$ being now functions of $Q$, one can show that $\mathrm{c}_{\mathrm{HZ}}(U) \leq \pi r_1^2$ and, hence, $\mathrm{c}_{\mathrm{HZ}}(U) = \pi r_1^2$.

Note now that $\mathrm{c}_{\mathrm{HZ}}(V) = \sup_U \mathrm{c}_{\mathrm{HZ}}(U)$, where the supremum is taken over all $U \subset V$ with compact closure. (The same is true for relative capacities, as long as $Z \subset U$.) Applying this to an exhaustion of a symplectic cylinder by ellipsoids, we obtain that $\mathrm{c}_{\mathrm{HZ}}(B_r^{2n} \times \mathbb{R}^{2m}) = \pi r^2$, where $B_r^{2n}$ is the ball of radius $r > 0$.

# Chapter 4

# Hamiltonian systems without periodic orbits

The first four sections of this chapter are devoted to the proof of Theorem 2.3.1. The proof is organized as follows. In Section 4.1 we describe the symplectic embedding of Schweitzer's flow assuming the existence of the plug with required properties. In Sections 4.2 and 4.3 we derive the existence of such a flow on the plug from the fact (Lemma 4.3.2) that there exists a "sufficiently smooth" Denjoy flow on $T^2$. Finally, this "sufficiently smooth" Denjoy flow is constructed in Section 4.4.

In Section 4.5 we construct functions with a sequence of regular levels without periodic orbits, converging to an isolated, but very degenerate, minimum.

## 4.1 Proof of Theorem 2.3.1: the symplectic embedding

Let us first fix some notations. Throughout this paper $\sigma$ denotes the standard symplectic form on $\mathbb{R}^{2m}$ or the pull-back of this form to $\mathbb{R}^{2m+1}$ by the projection $\mathbb{R}^{2m+1} \to \mathbb{R}^{2m}$ along the first coordinate; $I^{2m}$ stands for a cube in $\mathbb{R}^{2m}$ whose edges are parallel to the coordinate axis. The product $[a,b] \times I^{2m}$ is always assumed to be embedded into $\mathbb{R}^{2m+1}$ (henceforth, the standard embedding) so that the interval $[a,b]$ is parallel to the first coordinate. We refer to the direction along the first coordinate $t$ (time) in $\mathbb{R}^{2m+1}$ (or $[a,b]$ in $[a,b] \times I^{2m}$) as the vertical direction.

All maps whose smoothness is not specified are $C^\infty$-smooth.

Theorem 2.3.1, as do similar theorems in dimensions greater than four, follows from the existence of a symplectic plug. The definitions of a plug vary considerably (see [Gi1, Ke2, KuG]), and here we use the one more suitable for our purposes.

A *$C^k$-smooth symplectic plug* in dimension $2n$ is a $C^k$-embedding $J$ of $P = [a,b] \times I^{2n-2}$ into $P \times \mathbb{R} \subset \mathbb{R}^{2n}$ such that

P1. *The boundary condition*: The embedding $J$ is the identity embedding of $P$ into $\mathbb{R}^{2n-1}$ near the boundary $\partial P$. Thus the characteristics of $J^*\sigma$ are parallel to the vertical direction near $\partial P$.

P2. *Aperiodicity*: The characteristic foliation of $J^*\sigma$ is aperiodic, i.e., $J^*\sigma$ has no closed characteristics.

P3. *Existence of trapped trajectories*: There is a characteristic of $J^*\sigma$ beginning on $\{a\} \times I^{2n-2}$ that never exits the plug. Such a characteristic is said to be trapped in $P$.

P4. The embedding $J$ is $C^0$-close to the standard embedding and $C^k$-isotopic to it.

P5. *Matched ends or the entrance–exit condition*: If two points $(a,x)$, the "entrance", and

$(b, y)$, the "exit", are on the same characteristic, then $x = y$. In other words, for a characteristic that meets both the bottom and the top of the plug, its top end lies exactly above the bottom end.

**Theorem 4.1.1.** *In dimension four, there exists a $C^2$-smooth symplectic plug.*

*Proof of Theorem 2.3.1.* Theorem 2.3.1 readily follows from Theorem 4.1.1. Consider an irrational ellipsoid in $\mathbb{R}^4$ and pick two little balls each of which is centered at a point on a closed characteristic on the ellipsoid. Intersections of these balls with the ellipsoid can be viewed symplectically as open subsets in $\mathbb{R}^3$. By scaling the plug we can assume that $[a, b] \times I^2$ can be embedded into each of these open balls so that the closed characteristic on an ellipsoid matches a trapped trajectory in the plug. Now we perturb the ellipsoid by means of the embedding $J$ within each of these open subsets. The resulting embedding has no closed characteristics, $C^0$-close to the ellipsoid and $C^2$-isotopic to it. □

*Proof of Theorem 4.1.1.* First observe that it suffices to construct a semi-plug, i.e., a "plug" satisfying only the conditions (P1)-(P4). Indeed, a plug can then be obtained by combining two symmetric semi-plugs. More precisely, suppose that a semi-plug with embedding $J_-$ has been constructed. Without loss of generality we may assume that $[a, b] = [-1, 0]$. Define a semi-plug on $[0, 1] \times I^2$ with embedding $J_+$ by setting $J_+(t, x) = RJ_-(-t, x)$, $t \in [0, 1]$ and $x \in I^2$, where $R$ is the reflection of $\mathbb{R}^4$ in $\mathbb{R}^3$. Combined together, these semi-plugs give rise to a plug on $[-1, 1] \times I^2$.

We will construct a semi-plug by perturbing the standard embedding of $[a, b] \times I^2$ on a subset $M \subset [a, b] \times I^2$. This subset is diffeomorphic to $[-1, 1] \times \Sigma$, where $\Sigma$ is a punctured torus.

It is more convenient to do this perturbation using slightly different "coordinates" on a neighborhood of $M$. More specifically, we will first consider an embedding of $M$ into another four-dimensional symplectic manifold $(W, \sigma_W)$ such that the pull-back of $\sigma_W$ is still $\sigma|_M$. Then we $C^0$-perturb this embedding so that the characteristic vector field of the new pull-back will have properties similar to those of Schweitzer's plug. By the symplectic neighborhood theorem, a neighborhood of $M$ in $W$ is symplectomorphic to that of $M$ in $\mathbb{R}^4$. This will allow us to turn the embedding $M \hookrightarrow W$ into the required embedding $J\colon M \hookrightarrow \mathbb{R}^4$. (See the diagrams (4.1.1) and (4.1.2) below.)

To construct the perturbed embedding $M \hookrightarrow W$, we first embed $M$ into $[-1,1] \times T^2$ by puncturing the torus in a suitable way. Then we find a map $j\colon [-1,1] \times T^2 \to W$ such that the characteristic vector field of $j^*\sigma_W$ is aperiodic and has trapped trajectories.

The embedding $j$ is constructed as follows. Let $(x, y)$ be coordinates on $T^2$. Consider the product $W = (-2,2) \times S^1 \times T^2$ with coordinates $(t, x, u, y)$ and symplectic form $\sigma_W = dt \wedge dx + du \wedge dy$. The map $j$ is a $C^0$-small perturbation of

$$j_0\colon [-1,1] \times T^2 \to W; \quad j_0(t,x,y) = (t,x,x,y).$$

Note that $j_0(t,x,y) = (t,x,K_0,y)$, where $K_0(t,x,y) = x$. To define $j$, let us replace $K_0$ by a mapping $K\colon [-1,1] \times T^2 \to S^1$ to be specified later on. In other words, set

$$j\colon [-1,1] \times T^2 \to (-2,2) \times S^1 \times T^2, \quad \text{where} \quad j(t,x,y) = (t,x,K,y).$$

It is clear that $j$ is an embedding. (An explanation of the origin of $j$ is given in Remark 4.1.4.) The pull-back $j^*\sigma_W$ is the form

$$j^*\sigma_W = dt \wedge dx + (\partial_x K) dx \wedge dy + (\partial_t K) dt \wedge dy$$

with characteristic vector field

$$v = (\partial_x K)\partial_t - (\partial_t K)\partial_x + \partial_y.$$

To ensure that (P1)-(P4) hold we need to impose some requirements on $K$.

To specify these requirements, consider a Denjoy vector field $\partial_y + h\partial_x$ on $T^2$. This vector field should satisfy certain additional conditions which will be detailed in Section 4.4. Denote by $\mathfrak{D}$ the Denjoy continuum for this field. (Recall that $\mathfrak{D}$ is the closure of a trajectory of the Denjoy vector field; see Section 4.4.1 for the precise definition.)[1] Pick a point $(x_0, y_0)$ in the complement of $\mathfrak{D}$. Fix a small, disjoint from $\mathfrak{D}$, neighborhood $V_0$ of $(x_0, y_0)$. Consider the tubular neighborhood of the line $(t, x_0, y_0 + t)$ in $[-1, 1] \times T^2$ of the form $\{(t, x, y + t) \mid (x, y) \in V_0,\ t \in [-1, 1]\}$. Fix also a small neighborhood of the boundary $\partial([-1, 1] \times T^2)$ and denote by $N$ the union of these neighborhoods.

**Proposition 4.1.2.** *There exists a $C^2$-smooth mapping $K \colon [-1, 1] \times T^2 \to S^1$ such that*

- *K1. $v$ is equal to the Denjoy vector field (i.e., $\partial_x K = 0$ and $\partial_t K = -h$) at every point of $\{0\} \times \mathfrak{D}$;*
- *K2. the $t$-component of $v$ is positive (i.e., $\partial_x K > 0$) on the complement of $\{0\} \times \mathfrak{D}$;*
- *K3. $K$ is $C^0$-close[2] to the map $K_0 \colon (t, x, y) \mapsto x$;*
- *K4. $K = K_0$ on $N$.*

Let us defer the proof of the proposition to Section 4.2 and finish the proof of Theorem 4.1.1. From now on we assume that $K$ is as in Proposition 4.1.2.

By (K1) and (K2), $v$ has a trapped trajectory and is aperiodic. Indeed, by (K1), $\{0\} \times \mathfrak{D}$ is invariant under the flow of $v$ and on this set the flow is a Denjoy flow. By (K2), the vertical component of $v$ is non-zero unless the point is in $\{0\} \times \mathfrak{D}$. This implies that periodic orbits can only occur within $\{0\} \times \mathfrak{D}$. Since the Denjoy flow is aperiodic, so is the entire flow

[1] We also refer the reader to [HS, KH, Schwe] for a discussion of Denjoy maps and vector fields.

[2] More specifically, for any $\epsilon > 0$ there exists $K$ satisfying (K1)-(K2) and (K4) such that $\| K - K_0 \| < \epsilon$. The required value of $\epsilon$ is determined by the size of the neighborhood $U$ in the symplectic neighborhood theorem; see below.

of $v$. Furthermore, it is easy to see that since $\{0\} \times \mathfrak{D}$ is invariant, there must be a trapped trajectory. Furthermore, $v = \partial_t + \partial_y$ on $N$ by (K4).

Now we are in a position to define $J$. Let $\Sigma$ be the torus $T^2$, punctured at $(x_0, y_0)$. To be more accurate, $\Sigma$ is obtained by deleting a neighborhood of $(x_0, y_0)$, contained in $V_0$. There exists a symplectic bridge immersion of $(\Sigma, dx \wedge dy)$ into some cube $I^2$ with the standard symplectic structure. Hence, there exists an embedding

$$M = [-1, 1] \times \Sigma \hookrightarrow [a, b] \times I^2 \subset \mathbb{R}^3 \subset \mathbb{R}^4$$

such that the pull back of $\sigma$ is $dx \wedge dy$. Henceforth, we identify $M$ with its image in $\mathbb{R}^4$.

On the other hand, we can embed $M$ into $[-1, 1] \times T^2$ by means of

$$\varphi \colon M = [-1, 1] \times \Sigma \to [-1, 1] \times T^2; \quad \varphi(t, x, y) = (t, x, y + t).$$

Then $\varphi_* \partial_t = \partial_t + \partial_y$ and $(j_0 \varphi)^* \sigma_W = dx \wedge dy$. The argument similar to the proof of the symplectic neighborhood theorem, [McDS, Lemma 3.14], shows (see [Gi1, Section 4] for details) that a "neighborhood" of $M$ in $\mathbb{R}^4$ is symplectomorphic to a "neighborhood" $U$ of $j_0 \varphi(M)$ in $W$. More precisely, for a small $\delta > 0$, there exists a symplectomorphism

$$\psi \colon M \times (-\delta, \delta) \to U \subset W$$

extending $j_0 \varphi$, i.e., such that $\psi|_M = j_0 \varphi$. These maps form the following diagram:

$$\begin{array}{ccccc} M & \hookrightarrow & (-\delta, \delta) \times M & \subset & \mathbb{R}^4 \\ \| & & \downarrow \psi & & \\ M & \xrightarrow{j_0 \varphi} & U & \subset & W \end{array} \tag{4.1.1}$$

By (K3), $j$ is $C^0$-close to $j_0$. Furthermore, $j = j_0$ on $N$ by (K4). Hence, $j$ can be assumed to take values in $U$ (see Remark 4.1.3).

Finally, set

$$J = \psi^{-1} j \varphi$$

on $M$. In other words, $J$ is defined by the diagram:

$$\begin{array}{ccccc} M & \xrightarrow{J} & (-\delta,\delta)\times M & \subset & \mathbb{R}^4 \\ \| & & \downarrow\psi & & \\ M & \xrightarrow{j\varphi} & U & \subset & W \end{array} \tag{4.1.2}$$

Then $(J^*\sigma)|_M = (j\varphi)^*\sigma_W$. To finish the definition of $J$, we extend it as the standard embedding to $[a,b]\times I^2 \smallsetminus M$.

The characteristic vector field of $J^*\sigma$ is $\partial_t$ in the complement of $M$ and $(\varphi^{-1})_*v$ on $M$. Since $(\varphi^{-1})_*v = \partial_t$ near $\partial M$, these vector fields match smoothly at $\partial M$. It is clear that (P1) is satisfied. Since $v$ has a trapped trajectory and is aperiodic, the same is true for $(\varphi^{-1})_*v$, i.e., the conditions (P2) and (P3) are met. The condition (P4) is easy to verify. Hence, $J$ is indeed a semi-plug. $\square$

*Remark* 4.1.3. The following argument shows in more detail why $j$ can be assumed to take values in $U$. Let us slightly shrink $M$ by enlarging the puncture in $T^2$ and shortening the interval $[-1,1]$. Denote the resulting manifold with corners by $M'$. The shrinking is made so that $\partial M' \subset N$ and hence $M \smallsetminus M' \subset N$. It follows that $U$ contains a genuine neighborhood $U'$ of $j_0\varphi(M')$. Thus, if $K$ is sufficiently $C^0$-close to $K_0$, we have $j(\varphi(M')) \subset U'$. On $j_0\varphi(M\smallsetminus M')$, we have $K = K_0$ by (K5) and hence $j = j_0$. Therefore, $j(\varphi(M)) \subset U$.

*Remark* 4.1.4. The definition of the embedding $j$ can be explained as follows. Let us view the annulus $[-1,1]\times S^1$ with symplectic form $dt\wedge dx$ as a symplectic manifold and the product $[-1,1]\times T^2$ as the extended phase space with the $y$-coordinate being the time-variable. Then we can regard $K$ as a (multi-valued) time-dependent Hamiltonian on $[-1,1]\times S^1$. The embeddings $j_0$ and $j$ identify the coordinates $t$, $x$, and $y$ on $[-1,1]\times T^2$ with those on $W$. Hence, we can view $W$ as the further extended time-energy phase space with the cyclic energy-coordinate $u$. Then $j$ is the graph of the time-dependent Hamiltonian $K$ in the extended time-energy phase space $W$. Now it is clear that $v$ is just the Hamiltonian vector field of $K$.

*Remark* 4.1.5. In the proof of Proposition 4.1.2 we will not require the Denjoy continuum $\mathfrak{D}$ to have zero measure. As a consequence, the union of characteristics entirely contained in the semi-plug can have Hausdorff dimension two because this set is the image of $\mathfrak{D}$ by a $C^2$-smooth embedding.

## 4.2 Proof of Proposition 4.1.2

Recall that $\partial_y + h\partial_x$ is a Denjoy vector field on $T^2$ whose choice will be discussed later on and $\mathfrak{D}$ is the Denjoy continuum for this field. Recall also that $V_0$ is a small, disjoint from $\mathfrak{D}$, neighborhood of $(x_0, y_0)$. Fix a slightly larger neighborhood $V_1$ of $(x_0, y_0)$ which contains the closure of $V_0$ and is still disjoint from $\mathfrak{D}$. Let $\epsilon > 0$ be sufficiently small.

Proposition 4.1.2 is an immediate consequence of the following

**Proposition 4.2.1.** *There exists a $C^2$-smooth mapping $K\colon [-\epsilon, \epsilon] \times T^2 \to S^1$ which satisfies (K1)-(K3) and the requirement*

*K4′. $K = K_0$ for all $t$ and $(x, y)$ in the fixed neighborhood $V_1$ of $(x_0, y_0)$.*

*Proof of Proposition 4.1.2.* Let $K$ be as in Proposition 4.2.1. We extend this function to $[-1, 1] \times T^2$ as the linear combination $\phi(t)K(t, x, y) + (1 - \phi(t))x$, where $\phi$ is a bump function equal to 1 for $t$ close to 0 and vanishing for $t$ near $\pm\epsilon$. Note that this linear combination is well defined, as an element of the short arc connecting $K(t, x, y)$ and $x$, due to (K3). Clearly, the linear combination satisfies (K1)-(K3). If the range of $t$ for which $\phi(t) = 1$ is sufficiently small it also satisfies (K4). $\square$

*Proof of Proposition 4.2.1.*

*Step 1: The extension of $h$ to $[-1,1]\times T^2$.* Our first goal is to extend $h$ from $T^2$ to $H\colon [-1,1]\times T^2 \to \mathbb{R}$ smoothly and so that $\partial_x H - \partial_x h$ is of order one in $t$.

**Lemma 4.2.2.** *Assume that $\alpha$ is sufficiently close to 1 and $h$ is $C^{1+\alpha}$. Then there exists a $C^1$-function $H\colon [-1,1]\times T^2 \to \mathbb{R}$ such that*

*H1.* $H(0,x,y) = h(x,y)$;

*H2.* $\partial_x H(t,x,y) = \partial_x h(x,y) + o(t)$ *uniformly in* $(x,y)$;

*H3. the function* $\int_0^t H(\tau,x,y)\,d\tau$ *is* $C^2$ *in* $(t,x,y)$.

At this moment only the assertion of Lemma 4.2.2 is essential and we defer the proof of the lemma to Section 4.3.

*Remark* 4.2.3. Since $H$ is only $C^1$-smooth, the condition (H2) does not hold automatically. However, as is easy to see from the proof of the lemma, one can find an extension $H$ such that $\partial_x H(t,x,y) = \partial_x h(x,y) + o(t^k)$ for any given $k$ and (H3) still holds, provided that $\alpha$ is sufficiently close 1 (in fact, $k/(k+1) < \alpha < 1$).

*Step 2: The definition of $K$.* From now on we fix the extension $H$, but allow the interval $[-\epsilon,\epsilon]$, on which it is considered, vary. We will construct the function $K$ of the form

$$K(t,x,y) = \int_0^t [-H(\tau,x,y) + f(x,y)\tau]\,d\tau + A(x,y), \tag{4.2.1}$$

where the "constant" of integration $A$ and the correction function $f$ are chosen so as to make (K1)-(K3) and (K4′) hold. Note that $A$ is actually a function $T^2 \to S^1$, whereas $H$ and $f$ are real valued functions. The main difficulty in the proof below comes from the combination of the conditions (K1) and (K2).

*Step 3: The auxiliary functions $A$ and $f$.* Let us now specify the requirements the functions $A$ and $f$ have to meet.

**Lemma 4.2.4.** *There exist a $C^2$-function $A\colon T^2 \to S^1$ and $C^\infty$-function $f\colon T^2 \to \mathbb{R}$ satisfying the following conditions:*

*A1. $\partial_x A \geq \eta(\partial_x h)^2$ for some constant $\eta > 0$ and $\partial_x A$ vanishes exactly on the Denjoy set $\mathfrak{D}$;*

*A2. there exists an open set $U \subset T^2$, containing $\mathfrak{D}$, such that $U \cap V_1 = \emptyset$ and*

$$\partial_x A|_{T^2 \smallsetminus U} \geq const > 0, \tag{4.2.2}$$

$$\partial_x f|_U \geq 4\eta^{-1} + 2; \tag{4.2.3}$$

*A3. $A$ is a $C^0$-close to $(x, y) \mapsto x$;*

*A4. $A(x, y) = x$ for $(x, y) \in V_1$.*

This lemma will also be proved in Section 4.3.

*Remark* 4.2.5. More specifically, the condition (A3) means that for fixed $h$ and $V_1$ one can find $A$ arbitrarily $C^0$-close to $(x, y) \mapsto x$ and satisfying other requirements of the lemma.

*Step 4: The properties of $K$.* Let us now prove that $K$ given by (4.2.1), i.e.,

$$K = -\int_0^t H\, d\tau + \frac{t^2}{2} f + A,$$

satisfies the requirements of Proposition 4.2.1, provided that $\epsilon > 0$ is small enough and $A$ and $f$ are as in Lemma 4.2.4. The function $K$ is $C^2$. Indeed, the first term is $C^2$ by (H3). By Lemma 4.2.4, the next term is $C^\infty$ and the last term, $A$, is $C^2$.

Condition (K1) is obvious: $\partial_t K|_{t=0} = -H|_{t=0} = -h$ by (H1) and $\partial_x K|_{\{0\}\times\mathfrak{D}} = \partial_x A|_{\mathfrak{D}} = 0$ by (A1).

Let us now turn to (K2). We will first show that

$$\partial_x K = -\int_0^t \partial_x H\, d\tau + \frac{t^2}{2}\partial_x f + \partial_x A \geq 0 \tag{4.2.4}$$

and then prove that the equality occurs only on $\{0\} \times \mathfrak{D}$.

Assume first that $(x, y) \in U$. By (H2) and (4.2.3), we have

$$\partial_x K \geq \partial_x A - t\partial_x h + 2\eta^{-1}t^2 + \left(t^2 + o(t^2)\right).$$

Obviously,

$$t^2 + o(t^2) \geq 0 \tag{4.2.5}$$

for $(x, y) \in U$ and all $t \in [-\epsilon, \epsilon]$, provided that $\epsilon > 0$ is small. Hence, to verify (4.2.4), it suffices to show that

$$\partial_x A - t\partial_x h + 2\eta^{-1}t^2 \geq 0. \tag{4.2.6}$$

By (A1), this follows from

$$\eta(\partial_x h)^2 - t\partial_x h + 2\eta^{-1}t^2 \geq 0.$$

Here all the terms are non-negative except, maybe, $-t\partial_x h$. Hence, it suffices to prove that at least one of the following two inequalities holds:

$$\eta(\partial_x h)^2 - t\partial_x h \geq 0, \tag{4.2.7}$$

$$-t\partial_x h + 2\eta^{-1}t^2 \geq 0. \tag{4.2.8}$$

Inequality (4.2.7) holds if (but not only if)

$$|t| \leq \eta|\partial_x h| \tag{4.2.9}$$

and (4.2.8) holds if (but not only if)

$$|t| \geq \frac{\eta|\partial_x h|}{2}. \tag{4.2.10}$$

Clearly, at least one of the inequalities (4.2.9) and (4.2.10) holds. This proves (4.2.4) for $(x, y) \in U$.

Assume now that $(x, y) \in T^2 \setminus U$. Then, by (A2) or, more specifically, by (4.2.2),

$$\partial_x K = \partial_x A + O(t) > const + O(t) > 0,$$

when $\epsilon > 0$ is small. Thus (K2) holds for $(x, y) \in (T^2 \smallsetminus U)$.

To finish the proof of (K2) we need to show that for $(x, y) \in U$ the equality in (4.2.4) implies that $t = 0$ and $(x, y) \in \mathfrak{D}$. Thus, assume that $(x, y) \in U$ and $\partial_x K(t, x, y) = 0$. Then (4.2.5) and (4.2.6) must become equalities. The equality (4.2.5) is possible only when $t = 0$. Setting $t = 0$ in the equality (4.2.6), we conclude that $\partial_x A(x, y) = 0$ and hence $(x, y) \in \mathfrak{D}$ by (A1).

The condition (K3) follows from (A3). Indeed, if $\epsilon > 0$ is small, $K$ is $C^0$-close to $A$ which, in turn, is $C^0$-close to $K_0$ by (A3).

The condition (K4′) need not be satisfied for $K$. By (A3), on $[-\epsilon, \epsilon] \times T^2$ the function $K$ is $C^0$-close to $K_0$, provided that $\epsilon > 0$ is small. Moreover, by (A4), the function $\partial_x K$ is $C^0$-close to 1 and $\partial_y K$ is $C^0$-close to 0 on a neighborhood of $[-\epsilon, \epsilon] \times \text{closure}(V_1)$, for small $\epsilon > 0$. Now it is easy to see that (taking a smaller $\epsilon > 0$, if necessary) we can modify $K$ near and on $[-\epsilon, \epsilon] \times V_1$ so as to keep (K1)-(K3) and make the new function satisfy (K4′). Indeed, let $\phi \colon T^2 \to [0, 1]$ be a bump function equal to 1 on $V_1$ and 0 outside of a small neighborhood of $V_1$. Then the linear combination $x\phi + (1 - \phi)K$ still satisfies (K1)-(K3) and also (K4′) if $\epsilon > 0$ is small enough. Note that this linear combination is well defined due to (A3). □

## 4.3 Proofs of Lemmas 4.2.2 and 4.2.4

### 4.3.1 Proof of Lemma 4.2.2

For $t \neq 0$ and $(x, y) \in T^2$, set $x_\pm = x \pm t^s/2$ and $y_\pm = y \pm t^s/2$ and define

$$H(t, x, y) = \frac{1}{t^{2s}} \int_{y_-}^{y_+} \int_{x_-}^{x_+} h(\xi, \zeta)\, d\xi d\zeta,$$

where $s$ is an even positive integer to be specified later. Also, let $H(0, x, y) = h(x, y)$. In other words, $H(t, x, y)$ is obtained by averaging $h$ over the square with side $t^s$, centered at $(x, y)$.[3]

[3]This extension of $h$ by averaging is somewhat similar to the one from [KuG].

*Condition (H2):* First note that $H$ is obviously differentiable in $x$ and $y$ for every $t$. Furthermore, it is easy to see that $H$ satisfies (H2), i.e., $\partial_x H = \partial_x h + o(t)$, provided that

$$s\alpha > 1. \tag{4.3.1}$$

Indeed, since $h$ is continuous, we have

$$\partial_x H(t,x,y) = \frac{1}{t^{2s}} \int_{y_-}^{y_+} \big(h(x_+,\zeta) - h(x_-,\zeta)\big)\, d\zeta.$$

By the mean value theorem, $h(x_+,\zeta) - h(x_-,\zeta) = t^s \partial_x h(x_0,\zeta)$, where $x_0$ is some point in $[x_-, x_+]$, depending on $\zeta$. Since the distance between $(x,y)$ and $(x_0,\zeta)$ does not exceed $t^s/\sqrt{2}$ and $\partial_x h$ is $\alpha$-Hölder, we have $|\partial_x h(x_0,\zeta) - \partial_x h(x,y)| \leq const \cdot (t^s)^\alpha$ with $const$ independent of $(x,y)$. Hence,

$$\begin{aligned} |\partial_x H(t,x,y) - \partial_x h(x,y)| &\leq \frac{1}{t^s} \int_{y_-}^{y_+} \big|\partial_x h(x_0,\zeta) - \partial_x h(x,y)\big|\, d\zeta \\ &\leq const \cdot t^{s\alpha}, \end{aligned}$$

where $const$ is independent of $(x,y)$. This proves (H2), provided that (4.3.1) holds.

*$C^1$-smoothness of $H$:* We will show that $H$ is $C^1$, provided that $h$ is $C^1$ and $s > 1$. (Note that (4.3.1) implies that $s > 1$.) The proof essentially amounts to repeated applications of the mean value theorem. However, for the sake of completeness, we give a detailed argument below.

It is clear that $\partial_x H$ is continuous in all the variables for $t \neq 0$ and continuous in $x$ and $y$ for all $t$. Its continuity in $t$ at $t = 0$ follows from (H2). Moreover, it is easy to see that $\partial_x H \to \partial_x h$ as $t \to 0$ even if $h$ is just $C^1$. The same reasoning applies to $\partial_y H$.

It remains to show that $\partial_t H$ exists and is continuous. Again this is obvious for $t \neq 0$. Using the fact that $h$ is $C^1$, one can easily check that

$$|H(t,x,y) - h(x,y)| \leq const \cdot t^s.$$

This immediately implies that $\partial_t H \mid_{t=0}= 0$ when $s > 1$. Thus, to establish the continuity of $\partial_t H$ at $t = 0$, we need to prove that $\partial_t H \to 0$ uniformly in $(x, y)$ as $t \to 0$. A straightforward calculation shows that

$$\partial_t H(t,x,y) = \frac{s}{2t^{s+1}} \int_{y_-}^{y_+} \mathcal{J}_y(\zeta)\, d\zeta + \frac{s}{2t^{s+1}} \int_{x_-}^{x_+} \mathcal{J}_x(\xi)\, d\xi, \tag{4.3.2}$$

where

$$\mathcal{J}_y(\zeta) = h(x_+,\zeta) + h(x_-,\zeta) - \frac{2}{t^s} \int_{x_-}^{x_+} h(\xi,\zeta)\, d\xi$$

and, similarly,

$$\mathcal{J}_x(\xi) = h(\xi,y_+) + h(\xi,y_-) - \frac{2}{t^s} \int_{y_-}^{y_+} h(\xi,\zeta)\, d\zeta.$$

By the mean value theorem, we have

$$\begin{aligned} |\mathcal{J}_y(\zeta)| &= |h(x_+,\zeta) + h(x_-,\zeta) - 2h(x_0,\zeta)| \\ &= |\partial_x h(x_2,\zeta)(x_+ - x_0) - \partial_x h(x_1,\zeta)(x_0 - x_-)| \\ &\leq \big(|\partial_x h(x_2,\zeta)| + |\partial_x h(x_1,\zeta)|\big) \cdot t^s \\ &\leq const \cdot t^s, \end{aligned}$$

where $x_0$, $x_1$, and $x_2$ are some points in $[x_-, x_+]$, depending $\zeta$ and the constant can be taken independent of $(x, y)$.) As a consequence, the first term in (4.3.2) (whose absolute value is bounded by $const \cdot t^{s-1}$) goes to zero as $t \to 0$ if $s > 1$. A similar argument shows that the second term also goes to zero. Therefore, $\partial_t H \to 0$ as $t \to 0$ uniformly in $(x, y)$, and hence $\partial_t H$ is continuous.

*Condition (H3):* Let us now prove (H3), i.e., that

$$F = \int_0^t H\, d\tau$$

is $C^2$, under some additional constraints on $s$ and $\alpha$.

It is clear that $F$ is $C^1$. Moreover, the continuity of $\partial_x H$ and $\partial_y H$ implies that $\partial_x\partial_t F = \partial_x H = \partial_t\partial_x F$ and $\partial_y\partial_t F = \partial_y H = \partial_t\partial_y F$. Thus these partial derivatives exist and are continuous.

Furthermore, we claim that the second order partial derivatives of $F$ in $x$ and $y$ also exist and are continuous, provided that

$$s(1-\alpha) < 1. \tag{4.3.3}$$

Let us examine, for example, $\partial_x\partial_y F$. Note that $F|_{t=0} = 0$, and hence $\partial_x\partial_y F|_{t=0} = 0$. Thus we may assume that $t \neq 0$. Clearly,

$$\partial_x\partial_y F = \partial_x \int_0^t G(\tau, x, y)\, d\tau, \tag{4.3.4}$$

where

$$G(\tau, x, y) = \frac{1}{\tau^{2s}} \int_{x_-}^{x_+} (h(\xi, y_+) - h(\xi, y_-))\, d\xi.$$

We claim that in (4.3.4) the integration in $\tau$ and $\partial_x$ can be interchanged. Indeed,

$$\begin{aligned} \big|\partial_x G(\tau, x, y)\big| &= \frac{1}{\tau^{2s}} \big|(h(x_+, y_+) - h(x_-, y_+)) - (h(x_+, y_-) - h(x_-, y_-))\big| \\ &= \frac{1}{\tau^{s}} \big|\partial_x h(x_2, y_+) - \partial_x h(x_1, y_-)\big|. \end{aligned}$$

Here $x_1$ and $x_2$ are some points whose distance to $x$ does not exceed $\tau^s/2$ and the second equality follows from the mean value theorem. Using the fact that $\partial_x h$ is $\alpha$-Hölder, we obtain

$$|\partial_x G(\tau, x, y)| \leq \frac{const}{\tau^s} (\tau^s)^\alpha = \frac{const}{\tau^{s(1-\alpha)}},$$

where the constant is independent of $x$ and $y$. As a consequence, if $s(1-\alpha) < 1$, i.e., (4.3.3) holds, the integral $\int_0^t \partial_x G(\tau, x, y)\, d\tau$ converges absolutely and uniformly in $(x, y)$. Thus it follows from (4.3.3) that the derivative $\partial_x\partial_y F$ exists and

$$\partial_x\partial_y F = \int_0^t \partial_x G(\tau, x, y)\, d\tau. \tag{4.3.5}$$

In addition, this implies that

$$\partial_x \partial_y F \to 0 \quad \text{as } t \to 0 \text{ uniformly in } x \text{ and } y. \tag{4.3.6}$$

Let us prove now that $\partial_x \partial_y F$ is continuous. The above analysis shows that this derivative is everywhere continuous in $t$ and in $(x, y)$ at $t = 0$. Hence, we only need to verify its continuity in $(x, y)$ at $t \neq 0$. For $t \neq 0$, the integral (4.3.5) can be broken up into two parts: the integral over $[0, \delta]$ and the integral over $[\delta, t]$. By (4.3.6), the first part can be made arbitrarily small uniformly in $(x, y)$ by choosing $\delta > 0$ small. The second part is obviously continuous in $(x, y)$. This implies that $\partial_x \partial_y F$ is continuous in $(x, y)$.

Other partial derivatives of $F$ in $x$ and $y$ can be dealt with in a similar fashion. Hence, to ensure that (H2) and (H3) hold, it suffices to have $s$ and $\alpha$ satisfy (4.3.1) and (4.3.3) simultaneously. Obviously, for every $\alpha$ sufficiently close to 1, there exists an even positive integer $s$ satisfying these inequalities. This completes the proof of the lemma.

*Remark* 4.3.1. The assumption that $s$ is a positive even integer can be dropped if we replace $t^s$ by $|t|^s$ in the definition of $H$. Then (4.3.1) and (4.3.3) have a solution $s > 0$ if and only if $1/2 < \alpha < 1$.

### 4.3.2 Proof of Lemma 4.2.4

*Step 1: The function $A$.* The following lemma, which we will prove in Section 4.4, plays a crucial role in the construction of $A$.

**Lemma 4.3.2.** *There exists a Denjoy vector field $\partial_y + h\partial_x$ which is $C^{1+\alpha}$ for all $\alpha \in (0, 1)$ and such that*

*D1. $\partial_x h$ vanishes on $\mathfrak{D}$;*

*D2.* $\int_0^x (\partial_x h(\xi,y))^2\,d\xi$ *is* $C^2$ *in* $(x,y)$.

Assuming Lemma 4.3.2, let us continue the proof of Lemma 4.2.4. The essence of the requirements on $A$ is that $A$ should be $C^2$, and the derivative $\partial_x A$ should vanish on $\mathfrak{D}$ and be bounded from below by $\eta(\partial_x h)^2$. If $\eta(\partial_x h)^2$ were not sufficiently smooth, these conditions would be hard to satisfy. However, since $\int_0^x \eta(\partial_x h)^2\,d\xi$ is $C^2$ by Lemma 4.3.2, we may simply take $\eta(\partial_x h)^2$ as $\partial_x A$ with some additional correction terms. These extra terms are needed to make $A$ into a function $T^2 \to S^1$ meeting other requirements of Lemma 4.2.4.

Let us now outline the construction of $A$ omitting some details to be filled in at the concluding part of the proof (Step 3). Pick a smooth $C^1$-small non-negative function $a\colon T^2 \to \mathbb{R}$ which vanishes exactly on $\mathfrak{D}$. There exists a smooth non-negative function $b\colon T^2 \to \mathbb{R}$ which vanishes on $\mathfrak{D}$ (but not only on $\mathfrak{D}$) and such that

$$(x,y) \mapsto \int_0^x b(\xi,y)\,d\xi \quad \text{is } C^0\text{-close to} \quad (x,y)\mapsto x. \tag{4.3.7}$$

Pick a small $\eta_1 > 0$ and set[4]

$$A(x,y) = \frac{\int_0^x \left[\eta_1(\partial_x h)^2 + a + b\right]\,d\xi}{\int_0^1 \left[\eta_1(\partial_x h)^2 + a + b\right]\,d\xi}.$$

This is a function $T^2 \to S^1$. Indeed, $\int_0^1 \partial_x A\,dx = 1$ for any $y$ and $A(x,0) = A(x,1)$ for any $x$ by the definition of $A$. By (D2) and since $a$ and $b$ are smooth, $A$ is $C^2$. By taking $a$ and $\eta_1 > 0$ small, we can ensure that $A$ is $C^0$-close to $(x,y)\mapsto x$, i.e., the requirement (A3) is met. Also, $A$ obviously satisfies (A1) for some $\eta > 0$.

One can construct $b$ in such a way that $b \geq \mathit{const}$ on the complement of some neighborhood $U$ of $\mathfrak{D}$, which implies (4.2.2), and so that $b|_{V_1} = 1$. Then on $V_1$, the function $A$ is $C^1$-close to $x$, provided that $\eta_1 > 0$ is small and $a$ is $C^1$-small. Now it is easy to alter $A$ on and near $V_1$ so that (A4) is satisfied (i.e., $A(x,y) = x$ on $V_1$) and the conditions (A1) and (4.2.2) still hold.

[4] Throughout the proof we identify $T^2$ with $\mathbb{R}^2/\mathbb{Z}^2$.

*Step 2: The function $f$.* First note that it suffices to construct a function $f$ such that $\partial_x f|_U \geq const$. To define such a function $f$, we pick a smooth function $f'$ such that $f'|_U \geq const$ and such that the mean value of $x \mapsto f'(x,y)$ is zero for every $y$. (This is possible if $U$ is sufficiently small.) Then $f(x,y) = \int_0^x f'(\xi, y)\, d\xi$ satisfies (4.2.3).

*Step 3: The detailed construction of the neighborhood $U$ and the functions $b$ and $f$.* Let us cover $T^2$ by two open overlapping cylinders $C_1 = S^1 \times I_1$ and $C_2 = S^1 \times I_2$, where $I_1$ and $I_2$ are two arcs covering the circle $S^1$ with coordinate $y$.

First we describe $b$ and $f$ on $C_1$. For the sake of brevity let us denote $C_1$ by $C$ and $I_1$ by $I$. Without loss of generality we may assume that $0 \in I$ and $V_1 \subset C$. The Denjoy flow gives rise to a $C^1$-diffeomorphism $\varphi \colon C \to S^1 \times I$ which sends $\mathfrak{D} \cap C$ to a cylindrical set, i.e., $\varphi(\mathfrak{D} \cap C) = \mathfrak{D}_0 \times I$, where $\mathfrak{D}_0 = \mathfrak{D} \cap \{y = 0\}$. It is easy to see that $\mathfrak{D}_0$ can be covered by a finite collection of disjoint arbitrarily short open intervals $\Gamma_1, \ldots, \Gamma_k$. Then $\varphi(\mathfrak{D} \cap C)$ is covered by stripes $\Gamma_i \times I$ and thus $\mathfrak{D} \cap C$ is covered by the skewed stripes $\varphi^{-1}(\Gamma_i \times I)$.

The intersection of $\varphi^{-1}(\Gamma_i \times I)$ with $S^1 \times \{y\}$ is an arc whose end-points are $C^1$-functions of $y$. For each stripe, let us approximate these functions by $C^\infty$-functions. If the approximations are accurate enough, the new end-point functions still bound non-overlapping skewed open stripes in $C$ which cover $\mathfrak{D} \cap C$. Denote these stripes by $L_1, \ldots, L_k$ and set $U_1 = \cup L_i$.

Note that the end-points of $L_i \cap (S^1 \times \{y\})$ are smooth functions of $y$ and that all $L_i$ can be made arbitrarily narrow by taking the intervals $\Gamma_i$ short. In addition, we can always make $U_1$ disjoint from $V_1$.

It is not hard to see that there exists a $C^\infty$-function $b$ on $C$ which is identically zero on $U_1$ and such that (4.3.7) holds. Indeed, on $S^1 \times \{y\}$ we take a smooth function which is equal to zero on all arcs $L_i \cap (S^1 \times \{y\})$ and has high bumps in between these arcs. Since the arcs are short, $b$ can be chosen to satisfy (4.3.7). This function can be obviously made smooth in $y$ because so are the end-points of the arcs. In addition, it is easy to see that we can take $b$ to be equal to 1 on $V_1$.

The function $f$ is defined in a similar fashion. For example, we can take $f'$ equal to 1 on $U_1$ and, for each $y$, and use the complement of $U_1 \cap (S^1 \times \{y\})$ in $S^1 \times \{y\}$ to make sure that $f'$ has zero mean. Then, as we have pointed out above, we set $f(x,y) = \int_0^x f'(\xi, y)\, d\xi$.

For the second cylinder $C_2$ the argument is similar. The function $b$ on $T^2$ is obtained from its counterparts $b_1$ on $C_1$ and $b_2$ on $C_2$ by pasting $b_1$ and $b_2$ on $C_1 \cap C_2$ using cut-off functions in $y$. The construction of $f$ is finished in a similar way. It is clear that there exists a small neighborhood $U$ of $\mathfrak{D}$ (contained in $U_1 \cup U_2$) such that $b|_U = 0$ and $\partial_x f|_U = 1$. This is the required neighborhood $U$. (Note that in general we cannot take $U = U_1 \cup U_2$.) The proof of the lemma is completed.

## 4.4 Proof of Lemma 4.3.2

The proof of Lemma 4.3.2 is based on the existence of a $C^{1+\alpha}$ Denjoy diffeomorphism $\Phi$ such that $(\Phi' - 1)^2$ is $C^1$. Therefore, we first outline the construction of such a Denjoy diffeomorphism, and then proceed with the proof of the lemma.

### 4.4.1 Construction of the Denjoy diffeomorphism

**Lemma 4.4.1.** *There exists a Denjoy diffeomorphism $\Phi$ which is $C^{1+\alpha}$ for all $\alpha \in (0,1)$ and such that $(\Phi' - 1)^2$ is $C^1$.*

*Proof of Lemma 4.4.1.* We prove Lemma 4.4.1 in two steps. First, we define the required Denjoy diffeomorphism $\Phi$ and show that $\Phi'$ is $\alpha$-Hölder for every $\alpha \in (0,1)$, then we prove that $(\Phi' - 1)^2$ is $C^1$.

*Step 1: Definition of $\Phi$.* In the construction of $\Phi$ we closely follow the general description of Denjoy maps in [KH, Section 12.2]. Pick $\beta \in (0,1)$ and let

$$l_n := k_\beta (|n|+2)^{-1} (\log (|n|+2))^{-1/\beta} \tag{4.4.1}$$

be the length of the interval $I_n$ inserted into $S^1$ to "blow up" an orbit, $a_n$, of an irrational rotation. Here $k_\beta$ is a constant depending on $\beta$ chosen so that $\sum_{n\in\mathbb{Z}} l_n < 1$. We emphasize that this choice of $l_n$ is essential in order to make the series $\sum_{n\in\mathbb{Z}} l_n$ converge very slowly which, in turn, results in a small Denjoy continuum, $S^1 \smallsetminus \bigcup_{n\in\mathbb{Z}} Int(I_n)$. This slow convergence is the main factor which ensures that $(\Phi'-1)^2$ is $C^1$ and the second assertion (D2) of Lemma 4.3.2 holds, as it will become clear later on.

To construct a Denjoy diffeomorphism $\Phi$, it suffices to define the derivative $\Phi'$, since $\Phi$ is then obtained by integration. Let $\varphi\colon [0,1] \to \mathbb{R}$ be a bump function satisfying $\int_0^1 \varphi(x)\,dx = 1$. Define the smooth function

$$\varphi_n(x) := c_n\, \varphi\big((x-a_n)/l_n\big)$$

on the interval $I_n = [a_n, a_n + l_n]$, where $c_n = (l_n - l_{n+1})/l_n$, and note that $\int_{I_n} \varphi_n(x)\,dx = c_n\, l_n$. Finally, let

$$\Phi'(x) = \begin{cases} 1 & \text{for } x \in S^1 \smallsetminus \bigcup_{n\in\mathbb{Z}} I_n, \\ 1+\varphi_n(x) & \text{for } x \in I_n. \end{cases}$$

This completes the construction of $\Phi$. It is well known, [KH], and easy to see that $\Phi$ is $C^{1+\alpha}$ for any $\alpha \in (0,1)$. (Moreover, one can show that $|\Phi'(x) - \Phi'(x_0)| \leq const|x-x_0|\big|\log|x-x_0|\big|^{1/\beta}$ for any $x$ and $x_0$ in $S^1$.) For what follows, we only need that $\Phi$ is $C^{1+\alpha}$ for some $\alpha \in (1/2, 1)$ and also some estimates on the $C^1$-norm of $(\Phi'-1)|_{I_n}$ which result from (4.4.1).

Let us now list some properties of $c_n$ and $l_n$ that we will use later on:

First, we note that, as is true for any Denjoy map,

$$c_n \to 0 \quad \text{as} \quad n \to \infty.$$

In fact, $c_n = O(1/n)$.

Furthermore, (4.4.1) guarantees[5] that

$$\frac{c_n^2}{l_n} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.4.2}$$

Indeed,

$$\frac{|c_n|}{l_n} = \frac{|l_n - l_{n+1}|}{l_n^2} = O\bigg(\big(\log\,(|n|+2)\,\big)^{1/\beta}\bigg),$$

as can be seen by expanding the left hand side in $|n|^a\left(\log\,(|n|+2)\,\right)^b$ for $a \leq 0$, and $b \geq 0$. Thus,

$$\frac{c_n^2}{l_n} = l_n\,\frac{c_n^2}{l_n^2} = \frac{O\big(\big(\log\,(|n|+2)\big)^{1/\beta}\big)}{(|n|+2)} \to 0 \quad \text{as} \quad n \to \infty.$$

We finish this discussion by establishing the following estimates which will be used in the rest of the proof[6]:

$$\|(\Phi' - 1)\,|_{I_n}\| \;=\; O(|c_n|) \to 0 \tag{4.4.3}$$

$$\left\|\partial_x\,(\Phi' - 1)\,|_{I_n}\right\| \;=\; O(|c_n|/l_n) \tag{4.4.4}$$

$$\left\|\partial_x\,(\Phi' - 1)^2\,|_{I_n}\right\| \;=\; O(c_n^2/l_n) \to 0 \tag{4.4.5}$$

To prove these estimates, we first recall that $(\Phi' - 1)|_{I_n} = \varphi_n$. Then, since $\|\varphi_n\| = |c_n| \cdot \|\varphi\|$, we have (4.4.3). The second estimate, (4.4.4), is proved as follows:

$$\|\partial_x\,(\Phi' - 1)\,|_{I_n}\| = \|\partial_x \varphi_n\| \leq \|\varphi'\|\frac{|c_n|}{l_n}.$$

Finally, (4.4.5) is a consequence of the first two estimates and (4.4.2).

*Step 2: Proof that* $(\Phi' - 1)^2$ *is* $C^1$. Let $\mathfrak{D}_0 := S^1 \smallsetminus \bigcup_{n\in\mathbb{Z}} Int(I_n)$ denote the Denjoy continuum. Observe that, since on each $I_n$ the function $\varphi_n$ is smooth, $\Phi' - 1$ as well as $(\Phi' - 1)^2$ are also smooth on $I_n$. Hence, we need to prove that for $x_0 \in \mathfrak{D}_0$, $(\Phi' - 1)^2$ is differentiable, its derivative at $x_0$ is zero, and $\partial_x\,(\Phi' - 1)^2\,(x) \to 0$ as $x \to x_0$.

[5]This is the the main point in the proof where the specific choice of $l_n$ made above is essential.
[6]Throughout the rest of the proof $\|\;\|$ denotes the sup-norm on $I_n$.

Recall that $\Phi' - 1$ is $\alpha$-Hölder continuous with $\alpha > 1/2$ and $(\Phi' - 1)^2 \equiv 0$ on $\mathfrak{D}_0$. It readily follows that $(\Phi' - 1)^2$ is differentiable, and its derivative is zero on $\mathfrak{D}_0$.

To finish the proof, it remains to show that $\partial_x (\Phi' - 1)^2 (x) \to 0$ as $x \to x_0 \in \mathfrak{D}_0$. Let $x_k$ be a sequence in $S^1 \smallsetminus \mathfrak{D}_0$ converging to $x_0$. Since $\mathfrak{D}_0$ is nowhere dense, there exists a sequence of intervals, $I_{n_k}$, such that $x_k \in I_{n_k}$ for $k \in \mathbb{N}$. Then, by (4.4.5), $\left|\partial_x (\Phi' - 1)^2 (x_k)\right| \to 0$ as $k \to \infty$, and this, together with $\partial_x (\Phi' - 1)^2 (x_0) = 0$, proves the assertion. $\square$

*Remark* 4.4.2. The Denjoy map defined by (4.4.1) is essentially as smooth as a Denjoy map can be made, up to using functions growing slower than logarithms, e.g., iterations of logarithms. The next significant improvement in smoothness would be to have $\log \Phi'$ of bounded variation or satisfying the Zygmund condition which is impossible; see [HS, JS, KH].

Now we are in a position to prove Lemma 4.3.2 which asserts: *There exists a Denjoy vector field $\partial_y + h\partial_x$ which is $C^{1+\alpha}$ for all $\alpha \in (0,1)$ and such that*

D1. *$\partial_x h$ vanishes on $\mathfrak{D}$;*

D2. *$\int_0^x (\partial_x h(\xi, y))^2 \, d\xi$ is $C^2$ in $(x, y)$.*

To prove this lemma we show that the Denjoy vector field $\partial_y + h\partial_x$ on $T^2$ for $\Phi$ described above satisfies (D1) and (D2). The proof of (D1) is straightforward and based on the explicit formula for $h$. The proof of (D2) is divided into two parts. In the first part (Section 4.4.3), we show that $(\partial_x h)^2$ is $C^1$, which obviously means that $\partial_x \int_0^x (\partial_x h(\xi, y))^2 \, d\xi$ is $C^1$. In the second part (Section 4.4.4), we show that $\partial_y \int_0^x (\partial_x h(\xi, y))^2 \, d\xi$ is $C^1$. These two results imply (D2).

### 4.4.2 Explicit formula for $h$ and the proof of (D1).

First we give explicit formulas for $h$ and $\partial_x h$, and fix some notations. Let

$$\Phi_y(x) = (1 - \delta(y))\, x + \delta(y)\, \Phi(x)\,, \tag{4.4.6}$$

where $\delta\colon [0,1] \to [0,1]$ is a non-negative, increasing, smooth function which is 0 for $y$ close to 0 and 1 for $y$ close to 1. Then the $x$-component $h$ of a Denjoy vector field can be expressed as

$$h(x,y) = \left(\partial_y \Phi_y \circ \Phi_y^{\mathrm{inv}}\right)(x),$$

where the function $\Phi_y^{\mathrm{inv}}(x)$ is the inverse of $\Phi_y(x)$ in the $x$-variable.

Analyzing the smoothness of these functions, we first observe that $\Phi_y(x)$ is clearly $C^{1+\alpha}$ in $(x,y)$. Moreover, $\Phi_y(x)$ is $C^\infty$ for $x \notin \mathfrak{D}_0$. Furthermore, $\Phi_y^{\mathrm{inv}}(x)$ is also $C^{1+\alpha}$ in $(x,y)$. To see this note that by the implicit function theorem $\Phi_y^{\mathrm{inv}}(x)$ is $C^1$ in $(x,y)$ and

$$\partial_y \Phi_y^{\mathrm{inv}}(x) = -\frac{\partial_y \Phi_y\left(\Phi_y^{\mathrm{inv}}(x)\right)}{\partial_x \Phi_y\left(\Phi_y^{\mathrm{inv}}(x)\right)},$$

where the denominator is bounded away from zero. Now it readily follows that $\partial_y \Phi_y^{\mathrm{inv}}(x)$ is $C^\alpha$ in $(x,y)$, for the numerator is $C^{1+\alpha}$ and the denominator is $C^\alpha$. A similar argument shows that $\partial_x \Phi_y^{\mathrm{inv}}(x)$ is $C^\alpha$ in $(x,y)$.

As a consequence, $h$ is $C^{1+\alpha}$, and hence

$$\partial_x h(x,y) = \delta'(y)\left(\Phi' \circ \Phi_y^{\mathrm{inv}}(x) - 1\right)\partial_x \Phi_y^{\mathrm{inv}}(x)$$

is $C^\alpha$.

Finally, for a fixed $y \in [0,1]$, keeping the notation from Section 4.3.2, let $\mathfrak{D}_y := \mathfrak{D} \cap \{y\}$. Thus, $\mathfrak{D}_y = \Phi_y(\mathfrak{D}_0)$.

*Proof of (D1).* Let $(x,y) \in \mathfrak{D}$, i.e., $x \in \mathfrak{D}_y = \Phi_y(\mathfrak{D}_0)$. Thus, $\Phi_y^{\mathrm{inv}}(x) \in \mathfrak{D}_0$. Since $\Phi' - 1 \equiv 0$ on $\mathfrak{D}_0$, we conclude that $\partial_x h(x,y) = 0$, i.e., $\partial_x h$ vanishes on $\mathfrak{D}$.

### 4.4.3 Proof of (D2), Part I: $(\partial_x h)^2$ is $C^1$.

Note that the existence and continuity of the partial derivatives of $(\partial_x h)^2$ is non-trivial only at the points of $\mathfrak{D}$.

First, observe that both of the partial derivatives $\partial_x (\partial_x h)^2$ and $\partial_y (\partial_x h)^2$ exist and vanish at $(x, y) \in \mathfrak{D}$. This follows immediately from the facts that $\partial_x h$ is $\alpha$-Hölder continuous with $\alpha > 1/2$ and $\partial_x h$ vanishes on $\mathfrak{D}$ by (D1).

To examine the continuity of $\partial_x (\partial_x h)^2$ and $\partial_y (\partial_x h)^2$ we adopt a new notation for $(\partial_x h)^2$. Fix $y \in [0, 1]$ and let $F_y : S^1 \to \mathbb{R}$ be the function defined by

$$F_y(\xi) = \big(\Phi'(\xi) - 1\big)^2 \frac{\big(\delta'(y)\big)^2}{\big(1 + \delta(y)(\Phi'(\xi) - 1)\big)^2}. \tag{4.4.7}$$

Then

$$(\partial_x h(x, y))^2 = F_y \circ \Phi_y^{\mathrm{inv}}(x). \tag{4.4.8}$$

It follows that $F_y$ vanishes on $\mathfrak{D}_0$ for every $y$. The function $F_y$ is clearly differentiable since $F_y(\xi) = (\partial_x h)^2 (\Phi_y(\xi), y)$, where $(\partial_x h)^2$ is differentiable as is shown above and $\Phi_y(\xi)$ is $C^{1+\alpha}$ as proved in Section 4.4.2. Furthermore, $\partial_\xi F_y$ and $\partial_y F_y$ are both zero on $\mathfrak{D}_0$, for the partial derivatives of $(\partial_x h)^2$ vanish on $\mathfrak{D}$.

To prove that $(\partial_x h)^2$ is $C^1$ in $(x, y)$, it suffices to show that $F_y(\xi)$ is $C^1$ in $(\xi, y)$. (Indeed, $\Phi_y^{\mathrm{inv}}$ is $C^{1+\alpha}$ and (4.4.8) implies that $(\partial_x h)^2$ is $C^1$ if $F_y$ is $C^1$.) Thus, it remains to prove that $\partial_\xi F_y$ and $\partial_y F_y$ are continuous.

*Continuity of* $\partial_y F_y(\xi)$. This follows immediately from (4.4.7) since $\delta$ is $C^\infty$-smooth.

*Continuity of* $\partial_\xi F_y(\xi)$. First note that a straightforward calculation using (4.4.7) shows that

$$\partial_\xi F_y(\xi) = \big(\delta'(y)\big)^2 \frac{\big(1 + \delta(y)(\Phi' - 1)\big)\partial_\xi (\Phi' - 1)^2 - 2\,\delta(y)(\Phi' - 1)^2 \partial_\xi (\Phi' - 1)}{\big(1 + \delta(y)(\Phi' - 1)\big)^3}$$

on $S^1 \smallsetminus \mathfrak{D}_0 = \bigcup_{n \in \mathbb{Z}} Int(I_n)$, and, as discussed above, $\partial_\xi F_y(\xi) = 0$ on $\mathfrak{D}_0$. It follows immediately that $\partial_\xi F_y(\xi)$ is continuous in $y$ for every $\xi$.

Clearly, $\partial_\xi F_y(\xi)$ is continuous in $\xi$ on the complement of $\mathfrak{D}_0$ for every fixed $y$. Let us show the continuity at $(\xi, y)$ with $\xi \in \mathfrak{D}_0$. Note that the denominator in the expression for

$\partial_\xi F_y$ is bounded away from zero. Using the estimates (4.4.3), (4.4.4), and (4.4.5), it is easy to see that the asymptotic behavior of $\|\partial_\xi F_y|_{I_n}\|$ as $n \to \infty$ is determined by $\|\partial_\xi (\Phi' - 1)^2\|$, i.e.,

$$\left\|\partial_\xi F_y|_{I_n}\right\| = O(c_n^2/l_n) \to 0. \tag{4.4.9}$$

Arguing as in the proof of the fact that $(\Phi' - 1)^2$ is continuously differentiable (see Section 4.4.1), we conclude that $\partial_\xi F_y(\xi)$ is continuous.

This finishes the proof that $F_y$, and hence $(\partial_x h)^2$, is $C^1$.

### 4.4.4 Proof of (D2), Part II: $\partial_y \int_0^x (\partial_x h(\xi, y))^2 \, d\xi$ is $C^1$.

First let us write the function $\int_0^x (\partial_x h(\xi, y))^2 \, d\xi$ in a form more convenient for our analysis. Setting $\xi = \Phi_y(\eta)$, we obtain

$$\begin{aligned} \int_0^x (\partial_x h(\xi, y))^2 \, d\xi &= \int_0^x F_y\left(\Phi_y^{\mathrm{inv}}(\xi)\right) \, d\xi \\ &= \int_{\Phi_y^{\mathrm{inv}}(0)}^{\Phi_y^{\mathrm{inv}}(x)} F_y(\eta) \, \partial_\eta \Phi_y(\eta) \, d\eta. \end{aligned}$$

Hence, define

$$G_y(u) := \int_0^u F_y(\eta) \, \partial_\eta \Phi_y(\eta) \, d\eta. \tag{4.4.10}$$

Then

$$\begin{aligned} \int_0^x (\partial_x h(\xi, y))^2 \, d\xi &= \int_{\Phi_y^{\mathrm{inv}}(0)}^{\Phi_y^{\mathrm{inv}}(x)} F_y(\eta) \, \partial_\eta \Phi_y(\eta) \, d\eta \\ &= \int_0^{\Phi_y^{\mathrm{inv}}(x)} F_y(\eta) \, \partial_\eta \Phi_y(\eta) \, d\eta - \int_0^{\Phi_y^{\mathrm{inv}}(0)} F_y(\eta) \, \partial_\eta \Phi_y(\eta) \, d\eta \\ &= G_y \circ \Phi_y^{\mathrm{inv}}(x) - G_y \circ \Phi_y^{\mathrm{inv}}(0) \end{aligned}$$

Thus, our goal is to prove that $\partial_y[G_y \circ \Phi_y^{\mathrm{inv}}(x)]$ is a $C^1$-function. We do this in two steps: first, we show that the function $G_y(u)$ is $C^2$ and then, using this result, we prove that $\partial_y[G_y \circ \Phi_y^{\mathrm{inv}}(x)]$ is a $C^1$-function.

**Step 1: Proof that $G_y(u)$ is $C^2$.**

Let us show that both of the partial derivatives $\partial_u G_y(u)$ and $\partial_y G_y(u)$ are $C^1$.

*Proof that $\partial_u G_y(u)$ is $C^1$.* Let $\tilde{F}_y(u) := \partial_u G_y(u)$. Thus, by (4.4.10),

$$\tilde{F}_y(u) = F_y(u)\,\partial_u \Phi_y(u). \tag{4.4.11}$$

First, let us consider $\partial_y \partial_u G_y = \partial_y \tilde{F}_y$. For $u \in S^1 \setminus \mathfrak{D}_0$, the function $\tilde{F}_y(u)$ is smooth. Hence, as long as $u \in S^1 \setminus \mathfrak{D}_0$, the derivative $\partial_y \tilde{F}_y$ exists (and is continuous). For $u_0$ in $\mathfrak{D}_0$, the partial derivative $\partial_y \tilde{F}_y(u_0)$ exists and is zero. The reason is that $\tilde{F}_y(u_0) = 0$ for all $y \in [0,1]$. Furthermore, $\partial_y \tilde{F}_y(u)$ is continuous in $u$ and smooth in $y$, i.e., $\partial_y \tilde{F}_y(u)$ is infinitely differentiable in $y$ and every derivative is continuous in $(u, y)$ as immediately follows from (4.4.6) and (4.4.7). This proves the continuity of $\partial_y \partial_u G_y$.

Let us now focus on the partial derivative $\partial_u^2 G_y = \partial_u \tilde{F}_y$. As before, this partial derivative obviously exists when $u \in S^1 \setminus \mathfrak{D}_0$. Furthermore, we claim that $\partial_u \tilde{F}_y(u_0)$ exists and is zero for any $u_0 \in \mathfrak{D}_0$. To see this, recall that as we proved in Section 4.4.3, $F_y(u_0) = 0$ and $\partial_u F_y(u_0) = 0$ for all $u_0 \in \mathfrak{D}_0$. Hence,

$$\begin{aligned}
\partial_u \tilde{F}_y(u_0) &= \partial_u \big(F_y(u)\,\partial_u \Phi_y(u)\big)|_{u=u_0} \\
&= \lim_{u \to u_0} \frac{F_y(u)\,\partial_u \Phi_y(u) - \overbrace{F_y(u_0)}^{0}\,\partial_u \Phi_y(u_0)}{u - u_0} \\
&= \lim_{u \to u_0} \frac{F_y(u) - F_y(u_0)}{u - u_0}\,\partial_u \Phi_y(u) \\
&= \underbrace{\partial_u F_y(u_0)}_{0}\,\partial_u \Phi_y(u_0) \\
&= 0.
\end{aligned}$$

To show that $\partial_u \tilde{F}_y$ is continuous, we first express $\partial_u \tilde{F}_y$ on each $I_n$ as follows

$$\partial_u \tilde{F}_y(u) = \underbrace{\partial_u F_y(u)}_{O(c_n^2/l_n)}\,\underbrace{\partial_u \Phi_y(u)}_{O(1)} + \underbrace{F_y(u)}_{O(c_n^2)}\,\underbrace{\partial_u^2 \Phi_y(u)}_{O(|c_n|/l_n)},$$

where the braces indicate asymptotic behavior as $|n| \to \infty$. The estimate $\|\partial_u \tilde{F}\| = O(c_n^2/l_n)$ has been established above, see (4.4.9); the estimate $\|\partial_u \Phi_y\| = O(1)$ follows from the definition of $\Phi$ (see (4.4.6)) and (4.4.3); the estimate $\|F_y\| = O(c_n^2)$ is a consequence of the definition of $F_y$ (i.e., (4.4.7)) and (4.4.3). Finally, $\|\partial_u^2 \Phi_y\| = O(|c_n|/l_n)$ results from (4.4.6) and (4.4.4).

Now it is clear that $\|\partial_u \tilde{F}_y(u)_{I_n}\| = O(c_n^2/l_n)$. Since, by (4.4.2), $c_n^2/l_n \to 0$ as $|n| \to \infty$, $\partial_u \tilde{F}_y(u)$ can be shown to be continuous in a fashion similar to the cases discussed before.

This completes the proof of the fact that $\partial_u G_y(u) = \tilde{F}_y(u)$ is $C^1$.

*Proof that $\partial_y G_y(u)$ is $C^1$.* Note that, since $\partial_y \tilde{F}_y(\eta)$ is continuous in $(\eta, y)$ and its domain is compact, the functions $\partial_y \tilde{F}_y$ converge uniformly to $\partial_y \tilde{F}_y|_{y=y_0}$ as $y \to y_0$ for any $y_0 \in [0,1]$. Thus, we have

$$\partial_y G_y(u) = \partial_y \int_0^u \tilde{F}_y(\eta)\, d\eta = \int_0^u \partial_y \tilde{F}_y(\eta)\, d\eta.$$

This implies that $\partial_u \partial_y G_y(u)$ is continuous, for $\partial_u \partial_y G_y(u) = \tilde{F}_y(u)$ is continuous (in fact, $C^1$). To show that $\partial_y^2 G_y$ is continuous we recall that $\tilde{F}_y(u)$ is infinitely differentiable in $y$ and every derivative is continuous in $(u, y)$. Hence, as above, the integration and differentiation can be interchanged, and

$$\partial_y^2 G_y(u) = \int_0^u \partial_y^2 \tilde{F}_y(\eta)\, d\eta$$

is continuous because the integrand is contiunous. This completes Step 1.

**Step 2: Proof that $\partial_y[G_y \circ \Phi_y^{\text{inv}}(x)]$ is $C^1$.**

We first write this partial derivative explicitly as follows

$$\partial_y \big(G_y \circ \Phi_y^{\text{inv}}(x)\big) = \partial_u G_y\big(\Phi_y^{\text{inv}}(x)\big)\, \partial_y \Phi_y^{\text{inv}}(x) + \partial_y G_y\big(\Phi_y^{\text{inv}}(x)\big).$$

The second term of the sum on the right hand side is $C^1$ because it is the composition of $C^1$ and $C^{1+\alpha}$ functions. Thus, we focus on the first summand which is

$$\begin{aligned}\partial_u G_y\big(\Phi_y^{\mathrm{inv}}(x)\big)\,\partial_y\Phi_y^{\mathrm{inv}}(x) &= \tilde{F}_y\big(\Phi_y^{\mathrm{inv}}(x)\big)\,\partial_y\Phi_y^{\mathrm{inv}}(x)\\ &= F_y\big(\Phi_y^{\mathrm{inv}}(x)\big)\,\partial_u\Phi\big(\Phi_y^{\mathrm{inv}}(x)\big)\,\partial_y\Phi_y^{\mathrm{inv}}(x),\end{aligned}$$

where the last equality follows from (4.4.11). The product of the last two terms can be further simplified. Applying $\partial_y$ to the identity $\Phi_y\big(\Phi_y^{\mathrm{inv}}(x)\big) \equiv x$, we obtain

$$\partial_u\Phi_y\big(\Phi_y^{\mathrm{inv}}(x)\big)\,\partial_y\Phi_y^{\mathrm{inv}}(x) + (\partial_y\Phi_y)\big(\Phi_y^{\mathrm{inv}}(x)\big) = 0,$$

and hence

$$\tilde{F}_y\big(\Phi_y^{\mathrm{inv}}(x)\big)\,\partial_y\Phi_y^{\mathrm{inv}}(x) = -[F_y\,\partial_y\Phi_y]\circ\Phi_y^{\mathrm{inv}}(x).$$

Recall that $F_y(u)$, $\partial_y\Phi_y(u)$ and $\Phi_y^{\mathrm{inv}}(x)$ are all $C^1$-functions. Hence, the left hand side is also $C^1$.

This concludes Step 2 and hence the proof of the fact that $\partial_y\int_0^x(\partial_x h(\xi,y))^2\,d\xi$ is $C^1$.

*Remark* 4.4.3. Note that the norms of $(\Phi'-1)^2|_{I_n}$ and $\partial_u F_y(u)|_{I_n}$ and $\partial_u\tilde{F}_y(u)|_{I_n}$ converge to zero only as $O(c_n^2/l_n)$. (One can also show that the same is true for the $\partial_x$- and $\partial_y$-partial derivatives of $G_y\circ\Phi_y^{\mathrm{inv}}(x)$.) A faster rate of convergence, e.g., $O(|c_n|^3/l_n)$, would be likely to result in an "unacceptably" smooth Denjoy map and vector field.

## 4.5 Flows without periodic orbits in twisted cotangent bundles

In this section we prove Propositions 2.3.6, 2.3.8, and 2.3.9. Before getting into technical details of the proofs let us outline the basic line of reasoning in these constructions. All three arguments are similar concatenations of the following standard steps:

- Finding or creating, if not readily available, a function with a finite number of periodic orbits on a given level.

- Eliminating periodic orbits on this level.

- Applying the preceding two steps to a sequence of levels converging to zero.

- Smoothening up the resulting function at zero.

Hence, we describe the proofs with a varying degree of detail, focusing only on essential points.

### 4.5.1 Smoothing lemma

The last step is identical in all three proofs and we state it here as a lemma.

**Lemma 4.5.1.** *Let $Z$ be a closed submanifold of a manifold $W$ and let $F\colon W \to \mathbb{R}$ be a function such that*

1. *$F$ is non-negative, continuous, and vanishes on $Z$;*

2. *$F$ is $C^m$-smooth on the complement of $Z$ for some $0 \leq m \leq \infty$.*

*Then there exists a monotone increasing $C^\infty$-smooth function $\varphi\colon [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ and $\varphi'(y) > 0$ for $y > 0$, and such that $H = \varphi \circ F$ is $C^m$-smooth.*

*Remark* 4.5.2. We will need this lemma only in the cases where $m = 2$ and $m = \infty$. Below we prove it for $m = \infty$. The $m = 2$ case requires only obvious modifications.

*Proof.* [7] Fix a compact neighborhood $K$ of $Z$. Let $\varphi_l\colon [0,1] \to [0,\infty)$, $l = 0, 1, 2, \ldots$, be a sequence of smooth function such that

(S1) $\varphi_l(y) = 0$ when $0 \leq y \leq b_l^-$ and $\varphi_l(y) = c_l$ when $b_l^+ \leq y \leq 1$,

[7] The author is grateful to Antóny Serra for the proof of the lemma; [Se].

(S2) $\varphi_l'(y) > 0$ when $b_l^- < y < b_l^+$

for some intervals $(b_l^-, b_l^+) \subset (0,1)$ and constants $c_l > 0$, chosen so that the adjacent intervals overlap, $b_l^+ \to 0$, and

$$\| \varphi_l \|_{C^l([0,1])} \leq \frac{1}{2^l} \quad \text{and} \quad \| \varphi_l \circ F \|_{C^l(K)} \leq \frac{1}{2^l}. \tag{4.5.1}$$

The first condition in (4.5.1) guarantees that $\varphi = \sum \varphi_l$ is a smooth function on a neighborhood of zero, vanishing at zero, and such that $\varphi'(y) > 0$ for $y > 0$, by (S2) and since the intervals overlap. Let us extend this function from a neighborhood of zero to $[0, \infty)$ so that the resulting function $[0, \infty) \to [0, \infty)$, denoted again by $\varphi$, is smooth and still has these properties. Then, by the second condition of (4.5.1), $\varphi \circ F$ is smooth, i.e., $\varphi$ is the required function. $\square$

### 4.5.2 Proof of Proposition 2.3.6

We start with an irrational positive-definite quadratic form $G\colon \mathbb{R}^{2n} \to \mathbb{R}$. Clearly, every level $G = a$ has exactly $n$ distinct periodic orbits.

Assume first that $n > 2$ so that $\dim \mathbb{R}^{2n} > 4$. By inserting symplectic plugs as in [Gi1, Gi3, Gi4, He2, Ke2], we can eliminate periodic orbits on a sequence of levels $G = a_k$ with $a_k \to 0$. As a result, we obtain a function $F\colon \mathbb{R}^{2n} \to \mathbb{R}$ with the following properties:

- $F$ meets the requirements of Lemma 4.5.1 with $W = \mathbb{R}^{2n}$, $Z = \{0\}$, and $m = \infty$;
- $F$ attains its absolute minimum at zero;
- the levels $F = a_k$ carry no periodic orbits of the flow of $F$.

By Lemma 4.5.1, there exists a monotone increasing smooth function $\varphi\colon [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi'(y) > 0$ for $y > 0$ and such that $H = \varphi \circ F$ is $C^\infty$-smooth. Then it remains to set $\epsilon_k = \varphi(a_k)$.

When $n = 2$, the argument is similar, but the result of [GG1, GG2] is applied to eliminate periodic orbits. In this way we obtain a function $F$ which is only $C^2$-smooth on the complement of the zero section. As a consequence, the function $H$ is also only $C^2$-smooth. This completes the proof of Proposition 2.3.6.

### 4.5.3 Proof of Proposition 2.3.8

Here, we start with a non-symmetric Finsler metric $G\colon T^*S^n \to \mathbb{R}$ whose geodesic flow has a finite number of closed geodesics. A metric with this property has been constructed by A. Katok, [Ka]; see also [Zi]. The rest of the proof is identical to that of Proposition 2.3.6.

To deal with the case of the twisted geodesic flow for $\sigma \neq 0$, we observe that the flow of $G$ can be viewed as the twisted geodesic flow of the standard metric on $S^n$ for some exact non-zero magnetic field $\sigma$. The construction is finished in the same way as for the geodesic flow of $G$.

### 4.5.4 Proof of Proposition 2.3.9

The starting point of the construction is again a function $G$ whose flow has only finitely many periodic orbits on a given level or a sequence of levels converging to zero. The argument is particularly transparent when $n = 1$, i.e., for $M = \mathbb{CP}^1$.

**The construction for $T^*\mathbb{CP}^1$**

Let $g$ be the standard metric Hamiltonian on $W = T^*\mathbb{CP}^1 \to \mathbb{R}$ equipped with the twisted symplectic structure. Note that all orbits of the flow of $g$ on $W$ are periodic. Let $U$ be a neighborhood of a level $g = a$. The universal cover $\tilde{U}$ of $U$ is symplectomorphic to a neighborhood of a sphere $S^3$ in $\mathbb{C}^2$, centered at the origin. Let $\tilde{\Sigma}$ be an ellipsoid in $\tilde{U}$ which is close to $S^3$, invariant under deck transformations (the multiplication by $-1$), and carrying only finitely many closed characteristics. Then, $\tilde{\Sigma}$ descends to a hypersurface $\Sigma$ in $U$ with a finite number of closed characteristics, which is close to $g = a$. Note that $\tilde{\Sigma}$ can be taken arbitrarily close to $S^3$ and hence $\Sigma$ can be made arbitrarily close to $g = a$. Let now $\delta > 0$ be small and $U = g^{-1}((a - \delta, a + \delta))$. Then we can modify $g$ within $U$ so that the resulting function $G$ is isotopic to $g$ but has $\Sigma$ as the level $G = a$. The next step is eliminating, as above, all periodic orbits on the level $G = a$. The resulting function is now $C^2$-smooth.

Let us now pick a sequence $a_k \to 0$ and apply this process to each $a = a_k$. As a result, we obtain a continuous function $F$ which is $C^2$-smooth outside of the zero section and has the same properties as the functions $F$ constructed in the two previous proofs. As before, the proof for $n = 1$ is concluded by applying Lemma 4.5.1.

*Remark* 4.5.3. A. Katok's example of a Finsler metric on $S^2$ with only two closed geodesics can be easily described via a similar construction of the hypersurface $\Sigma$, starting with the level $g = 1$, in the standard $T^*S^2$, and then symplectically embedding its double cover into $\mathbb{C}^2$.

**The general case: $T^*\mathbb{CP}^n$**

Let $g_0$ be the standard metric Hamiltonian on $W = T^*\mathbb{CP}^n$. As before, all orbits of the flow of $g_0$ on $W$ are periodic and all orbits on a given level have the same period.

Fix a level $g_0 = a$. The first observation is that we can change $g_0$ in a neighborhood of $g_0 = a$ to a function of the form $g = f \circ g_0$, without altering the level sets of the function or creating critical points, so that the new function $g$ is equal to $g_0$ outside the neighborhood and its flow has constant period (say, equal to one) near the level $g_0 = a$. Without loss of generality, we may assume that $f(a) = a$, and hence the level we are working with is again $g = a$.

The next step is to notice that Ziller's method, [Zi], applies to the function $g$ near the level $g = a$, i.e., one can modify $g$ near the level to a $C^\infty$-function $G$, isotopic to $g$, so that the level $G = a$ carries only a finite number of periodic orbits.

For the sake of completeness, let us describe this modification in detail. As in [Zi], consider the action of $S^1 = \mathbb{R}/\mathbb{Z}$ on $\mathbb{CP}^n$ induced by the following diagonal $S^1$-action on $\mathbb{C}^{n+1}$:

$$t \cdot (z_0, \dots, z_n) = (e^{2\pi\lambda_0 t} z_0, \dots, e^{2\pi\lambda_n t} z_n), \tag{4.5.2}$$

where $\lambda_0, \dots, \lambda_n$ are mutually distinct integers and $\lambda_0 = 1$. This action preserves $\sigma$ and $g_0$ and hence the lift $\psi_t$ of this action to $T^*\mathbb{CP}^n$ preserves $g$ and the twisted symplectic structure $\omega$. The flow $\psi_t$ is Hamiltonian. Denote the Hamiltonian of $\psi_t$ by $g_1$ and set

$$G = g + \alpha g_1,$$

where $\alpha > 0$ is a small irrational number. Note that since $\alpha$ is small, the level $G = a$ lies near $g = a$ and, hence, is entirely contained in the region where the flow of $g$ has period one. We claim that the flow of $G$ on $T^*\mathbb{CP}^n$ has a finite number of periodic orbits on the level $G = a$ (and on nearby levels).

To prove this, let us first note that the closed orbits of the flow of $G$ near the level are in fact the closed orbits of $g$ which are invariant under the flow $\psi_t$. This can be easily checked by using the fact that the two flows commute and repeating word-for-word the reasoning from [Zi, p. 138]. It follows that the projections of these periodic orbits of $g$ to $\mathbb{CP}^n$ are invariant under the $S^1$-action. Next observe that every orbit of $g$ is a reparameterized orbit of the twisted geodesic flow on $g_0 = a$.

Hence, it suffices to show that the number of $S^1$-invariant twisted geodesics on $\mathbb{CP}^n$ coming from $g_0 = a$ is finite. Let $\gamma$ be a twisted geodesic. The initial conditions $(\gamma(0), \gamma'(0))$ determine a projective line $\mathbb{CP}^1 \subset \mathbb{CP}^n$ and $\gamma$ is tangent to this $\mathbb{CP}^1$. Furthermore, this projective line is invariant under the $S^1$-action if $\gamma$ is invariant. (These facts can be seen as follows. It is well known that $\gamma$ is the projection under the Hopf map of a great circle $\tilde{\gamma}$ in $S^{2n+1}$. The angle between $\tilde{\gamma}$ and the Hopf fibers is determined by $a$ and lies in the interval

$(0, \pi/2)$. Thus $\tilde{\gamma}(0)$ and $\tilde{\gamma}'(0)$ span a complex plane in $\mathbb{C}^{n+1}$. This complex plane gives rise to the projective line corresponding to $\gamma$ and is invariant under the action (4.5.2) if $\gamma$ is invariant.)

It is easy to see that there are only finitely many (in fact, $n(n+1)/2$) projective lines invariant under the action (4.5.2). (These projective lines arise from the $z_i z_j$-planes, $0 \le i < j \le n$.) On such a projective line, the twisted geodesic $\gamma$ is an (oriented) $S^1$-invariant spherical circle whose geodesic curvature is determined by $a$. There are only two such oriented circles.

Thus, as we have shown, the flow of $G$ on the level $G = a$ has a finite number of periodic orbits. Let us modify $G$ outside of a small neighborhood of $G = a$ so that the resulting function, which we denote by $G$ again, is $C^\infty$-smooth, equal to $g$ outside the neighborhood of $g = a$, and isotopic to $g$.

The proof is finished in the same way as for $n = 1$. Namely, as the next step, we eliminate all periodic orbits on the level $G = a$. This leads to a $C^\infty$-smooth (since $\dim T^*\mathbb{CP}^n > 4$) function which is again isotopic to $g$. Next, we apply this construction to a sequence $a_k \to 0$ of values of $g_0$ and obtain a continuous function $F$, $C^\infty$-smooth outside of the zero section and it remains to again utilize Lemma 4.5.1. This concludes the proof of Proposition 2.3.9 for all $n$.

*Remark* 4.5.4. The above argument also shows, along the lines of [Zi], that the flow of $G$ has exactly $n(n+1)$ periodic orbits on the level $G = a$.

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