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COUNTING PERIODIC ORBITS: CONLEY CONJECTURE FOR LAGRANGIAN CORRESPONDENCES AND RESONANCE RELATIONS FOR CLOSED REEB ORBITS

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Abstract

Counting Periodic Orbits:

Conley Conjecture for Lagrangian Correspondences and Resonance Relations for Closed Reeb Orbits

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The thesis is centered around the theme of periodic orbits of Hamiltonian systems. More precisely, we prove that on a closed symplectic Calabi–Yau manifold every Lagrangian correspondence Hamiltonian isotopic to the diagonal and satisfying some non-degeneracy condition has infinitely many periodic orbits, and we give a new proof of the theorem that every contact form supporting the standard contact structure on S^3 has at least two periodic Reeb orbits. The former result is obtained by considering the intersection Lagrangian Floer homology of suitable Lagrangians and estimating index growth for iterations, while the latter relies on a new homotopy invariant index which is in turn used to prove a new variant of the resonance relation for Reeb flows.

To my sweetheart Didem.

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Chapter 1

Introduction

In the realm of Hamiltonian systems, periodic orbits are considered to be among the most important objects of study. From celestial mechanics to electrodynamics, they model a variety of key phenomena in the physical world. Since the Hamiltonian dynamics had sprung out of the phase space description of physical problems, it is essential to consider Hamiltonians on a symplectic manifold and investigate the existence of their periodic orbits. Similarly, one can consider closed characteristics of a Hamiltonian on a closed energy surface, which gives rise to the existence problem for periodic Reeb orbits in contact manifolds. This thesis, in different chapters, is concerned with periodic orbits in these two separate settings.

1.1 Dynamics of Lagrangian Correspondences

The following is among the most prominent questions in symplectic geometry and Hamiltonian dynamics: Given a (time-dependent) Hamiltonian H :

$M \times \mathbb{R} \to \mathbb{R}$ on a closed symplectic manifold (M, ω), how many simple periodic orbits of all periods, if any, does its time-one map φ have? Such time-one maps are called *Hamiltonian diffeomorphisms*. While the existence part of this question is essentially answered affirmatively by the proof of the Arnold conjecture (see [54] and the references therein for a detailed discussion), the quantitative part of the problem (in a form of which we are interested in Chapter 2) is frequently referred as the Conley conjecture, which can be stated as follows:

> Given a (time-dependent) Hamiltonian $H : M \times \mathbb{R} \to \mathbb{R}$ on a symplectic manifold (M, ω) that satisfies some natural conditions, the time-one map φ has infinitely many simple periodic orbits. (CC)

Here, we call a k-periodic orbit $(\varphi^i(x))_{i \in \mathbb{Z}/k\mathbb{Z}}$ *simple* if all $\varphi^i(x)$ are distinct. Without any assumption on M, (CC) does not hold true: the rotation of the 2-sphere by an irrational angle has only two fixed points and no other simple periodic orbits. Nevertheless, the Conley conjecture is mostly true: For instance it holds for symplectic Calabi-Yau manifolds (i.e. $c_1(\mathrm{T}M) = 0$), surfaces, tori, negative monotone symplectic manifolds, etc. We refer the reader to [62, 17, 36, 25, 23, 34, 35] for various cases and [27] for a survey of recent results.

We are interested in a generalization of the Conley conjecture to Lagrangian correspondences. To briefly describe it here, we need to recall an important observation about symplectomorphisms made by Alan Weinstein in his seminal work [75]. Namely, for a symplectomorphism $\phi : (M, \omega_M) \to (N, \omega_N)$, its graph $\mathrm{Gr}(\phi) \subset M^- \times N$ is a Lagrangian submanifold with respect to the symplectic structure $(-\omega_M) \oplus \omega_N$. Therefore Lagrangian submanifolds of $M^- \times N$ are considered to be relations with additional structure, or, in other words, some generalization of symplectomorphisms. Such relations are called *Lagrangian correspondences.* Inherently, a Lagrangian correspondence $L \subset M^- \times M$, more conveniently denoted as $M \xrightarrow{L} M$, gives rise to a "dynamics" on M given by $x \overset{L}{\mapsto} y$ whenever $(x, y) \in L$. Since the domain and the codomain of L are both M, it is possible to define a (simple) periodic orbit as a sequence $(x_i)_{i \in \mathbb{Z}/k\mathbb{Z}}$, where $(x_i, x_{i+1}) \in L$ for all i (all x_i distinct). Coming back to (CC), note that for a Hamiltonian diffeomorphism $\varphi \colon M \to M$, its graph $\mathrm{Gr}(\varphi)$ gives rise to a Lagrangian correspondence $L = \mathrm{Gr}(\varphi)$ whose periodic orbits coincide with the periodic orbits of the Hamiltonian diffeomorphism. Moreover, the relation L is Hamiltonian isotopic to the diagonal $\Delta = \{(x, x) \mid x \in M\}$. Therefore, we can forget where L came from and just assume that we have a Lagrangian submanifold $L \subset M^- \times M$ Hamiltonian isotopic to the diagonal. Thus one can formulate the following generalization of Conley conjecture:

> Given a Lagrangian correspondence $L \subset M^- \times M$ Hamiltonian isotopic to the diagonal, where M satisfies some natural conditions, L has infinitely many simple periodic orbits. $\quad (\mathrm{CC}^{\mathrm{LC}})$

The expectation is, therefore, that this conjecture holds true in the cases of (CC) listed above. Chapter 2 is dedicated to proving $(\mathrm{CC}^{\mathrm{LC}})$ assuming that M is symplectic Calabi–Yau and L is, in a certain sense, weakly non-degenerate (see Definition 2.1.3 and Theorem 2.1.4 for details). Drawing a parallel with the Conley conjecture (for Hamiltonian diffeomorphisms), we can say that our result roughly corresponds to the case of (CC) proved by Salamon and Zehnder in [62].

To describe our approach used in the proof, let us consider the intersection Lagrangian Floer homology $HF(L,k) := HF(L^k, D^k)$ for $L^k = L \times \cdots \times L$ and $D^k = (\Delta \times \cdots \times \Delta)^T$, where both products are k-fold and $\cdot^T$ denotes the transposition of the last factor to the first. Notice that the intersection points of these two Lagrangians are (not necessarily simple) k-periodic points. We then proceed by analyzing the growth of the index of a fixed point under iterations. Combined with an isomorphism $HF_*(L,k) = H_{*+n}(M)$ (Theorem 2.4.2), the behavior of the index forces the correspondence to have infinitely many periodic orbits. This approach is similar to [62]; however, we employ a different version of "Maslov index theory" to compute indices of Lagrangian paths.

One important comment about the iterations is now due. The k^{th} iteration of a Hamiltonian diffeomorphism φ is simply the k-fold composition of φ with itself. Therefore, an iteration of a Hamiltonian diffeomorphism, or more precisely its graph, becomes a Lagrangian correspondence from M to itself. Also, the k-periodic points constitute precisely the intersection set $\mathrm{Gr}(\varphi^k) \cap \Delta$. However, for general Lagrangian correspondences $M \xrightarrow{L} N \xrightarrow{K} Q$, the set $K \circ L := \{(x,z) \in M^- \times Q \mid \exists y \in N, (x,y) \in L \text{ and } (y,z) \in K\}$ might not even be a smooth submanifold, and hence not necessarily a Lagrangian correspondence. Nevertheless, the k-periodic points of $M \xrightarrow{L} M$ are realized as the intersection set of two Lagrangians; more precisely, they are in one-to-one correspondence with $L^k \cap D^k$. This set belongs to $(M^- \times M)^k$, and both L^k and D^k are obviously smooth. Thus, in our approach, we bypass the smoothness problem in "composition of Lagrangian correspondences" (see [75, 74] for a detailed discussion about compositions) by considering the products mentioned above. Moreover $L^k \cap D^k$ is in one-to-one correspondence with the set $L^{\circ k} := L \circ \cdots \circ L$ (even if it is not a smooth submanifold), which makes our choice of Lagrangians more prevalent. Also, notice that even if the compositions are smooth (e.g. $L = \mathrm{Gr}(\varphi)$), computing the intersection Floer homology for arbitrary k and analyzing index growth are much more harder than they are in our approach.

1.2 Contact Dynamics and Resonance Relations

Another aspect of Hamiltonian dynamics we consider here concerns Reeb flows on contact manifolds. For completeness we shall revisit the basics of contact geometry now. A contact structure ξ is a maximally non-integrable hyperplane distribution on M. A contact form α for a coorientable distribution ξ is a 1-form such that $\ker \alpha = \xi$. The non-integrability condition translates as $\alpha \wedge (\mathrm{d}\alpha)^{n-1} \neq 0$ where $\dim M = 2n - 1$. The vector field R satisfying $\iota_R \, \mathrm{d}\alpha = 0$ and $\alpha(R) = 1$ is called the *Reeb vector field* for the contact form α and its integrals *Reeb orbits* (cf. [21] for a detailed introduction to contact geometry). Many interesting autonomous Hamiltonian flows (e.g. geodesic flows) are Reeb flows.

Notice that the linearized Poincaré return map restricted to ξ of a Reeb flow around a periodic orbit is symplectic, and thus has its mean index defined. When the number of closed Reeb orbits is finite, these mean indices must satisfy certain resonance relation. These relations for the standard contact sphere were found in [70], and later extended to non-degenerate Reeb flows for a broader class on contact manifolds in [29]. One of the main goals of Chapter 3 is to remove the non-degeneracy condition. We do this in several steps. To begin with, in Section 3.1, we prove an elementary formula relating the number of periodic orbits of an iterated map and the Lefschetz numbers of the iterations. We also establish a local version of this formula (see Section 3.1.2) which connects the number of periodic orbits, suitably defined, of a germ at an isolated fixed point and the indices of its iterations. The latter result is then used to express the mean Euler characteristic (MEC) discussed below, of a contact manifold with finitely many simple closed Reeb orbits in terms of local, purely topological, invariants of closed Reeb orbits, when the orbits are not required to be non-degenerate (see Section 3.2). This is the degenerate version of the resonance relations mentioned above. Finally, this relation is utilized to reprove a theorem asserting the existence of at least two Reeb orbits on the standard S^3 (see [11, 28, 50]) and the existence of at least two closed geodesics for a Finsler metric, not necessarily symmetric, on S^2 (see [2, 11]).

To describe our approach, let us consider first a smooth map $F\colon M \to M$, where M is a closed manifold. We show that the number $I_\kappa(F)$ of κ-periodic orbits of F, once the orbits of a certain type are discounted, can be expressed via the Lefschetz numbers of the iterations F^d for $d|\kappa$ (see Theorem 3.1.2), and hence $I_\kappa(F)$ is homotopy invariant. In a similar vein, given a germ of a smooth map at a fixed point x, isolated for all iterations, one can associate to it a certain invariant $I_\kappa(F)$, an iterated index, which counts κ-periodic orbits of a small perturbation of F near x, with again some orbits being discounted. The iterated index can also be expressed via the indices of the iterations F^d for $d|\kappa$ (see Theorem 3.1.4), and hence $I_\kappa(F)$ is again a homotopy invariant of F as long as x remains uniformly isolated for F^κ. We further investigate the properties of the iterated index in Section 3.1, which is essentially independent of the remainder of the chapter. The results in this section, although rather elementary, are new to the best of our knowledge; see however [10].

Next, we apply the iterated index to calculation of the mean Euler characteristic (MEC) of contact manifolds. The MEC of a contact manifold, an invariant introduced in [69] (cf. [59]), is the mean alternating sum of the dimensions of contact homology. It was observed in [29] that when a Reeb flow has finitely many simple periodic orbits and these orbits are totally non-degenerate, the MEC can be expressed via certain local invariants of the closed orbits, computable in terms of the linearized flow. (Here an orbit is called totally non-degenerate if all its iterations are non-degenerate.) This expression for the MEC generalizes resonance relations for the mean indices of the closed Reeb orbits on the sphere, proved in [70], and is further generalized to certain cases where there are infinitely many orbits in [15], including the Morse-Bott setting. The results of this type have been used for calculations of the MEC (see, e.g., [15]) and also in applications to dynamics; [29, 25, 59]. It is this latter aspect of the MEC formula that we are interested in here.

Versions of the MEC formula where the Reeb orbits are allowed to degenerate are established in [41, 51] with applications to dynamics in mind. In these formulas, however, the contributions of closed orbits are expressed in terms of certain local homology groups associated with the orbits and are contact-geometrical in nature. Here, in Theorem 3.2.2, we extend the result from [29] to the degenerate case with the orbit contributions computable via the linearized flow and a certain purely topological invariant of the Poincaré return map. The latter invariant is essentially the mean iterated index.

Two remarks concerning this result are due now. Firstly, in all known examples of Reeb flows, if there are finitely many periodic orbits, then they are totally non-degenerate. Thus the degenerate case of the local MEC formula (Theorem 3.2.2) appears to be of little interest for the MEC calculations for specific manifolds. However, it does have applications to dynamics. For instance, it allows one to rule out certain orbit patterns and, as a consequence, obtain lower bounds on the number of closed orbits (cf. [41, 50, 72]); see Section 3.3 and a discussion below. Secondly, our local MEC formula is identical to the one established in [41] and also, apparently, to the one proved for the sphere in [51]. The difference lies in the interpretation or the definition of the terms in the formula. Although our local MEC formula could be directly derived from [41, Theorem 1.5] (see Remark 3.2.6), we give, for the sake of completeness, a rather short proof of the formula, still relying, however, on some of the results from [41]; cf. [28]. In Section 3.2, we also discuss in detail the definition of the MEC, examples, the local and filtered contact homology and other ingredients of the proof of Theorem 3.2.2, and state a variant of the asymptotic Morse inequalities for contact homology (Theorem 3.2.7) generalizing some results from [29, 41].

Finally, in the last section, we turn to applications of Theorem 3.2.2. We prove in a novel way that every Reeb flow on the standard contact S^3 has at least two closed Reeb orbits; see [11, 28, 50][1]. The proofs in [28, 50] both rely on an analogue of the "degenerate case of the Conley conjecture" for contact forms, asserting that the presence of one closed Reeb orbit of a particular type (a so-called symplectically degenerate minimum) implies the existence of infinitely many closed Reeb orbits. This result holds in all dimensions; see [28]. Another non-trivial (and strictly three-dimensional) ingredient in the argument in [28] comes from the theory of finite energy foliations (see [37, 38]), while the argument in [50] uses, also in a non-trivial way, the variant of the local MEC formula from [51] for degenerate Reeb flows on the standard contact sphere. In this paper, we bypass the results from the theory of finite energy foliations and give a very simple proof of Theorem 3.3.1 utilizing Theorem 3.2.2 and the "Conley conjecture" type result mentioned above. The advantage of this approach is that it minimizes the 3-dimensional counterparts of the proof and, we hope, is the first step towards higher-dimensional results. We also reprove, in slightly more general form, a theorem from [2] that every Finsler metric on S^2 has at least two closed geodesics (Clearly, this also follows from [11]).

A word is due on the degree of rigor in Chapter 3, which varies considerably between its different parts. Section 3.1, dealing with the iterated index, is of course completely rigorous. The rest however, just as [28], heavily relies on the machinery of contact homology (see, e.g., [6, 13] and references therein), which is yet to be fully put on a rigorous basis (see [39, 40]). Note however that, to get around the

[1]In fact, a more general result is now known to be true. Namely, the assertion holds for any contact three-manifold. This fact, proved in [11] using the machinery of embedded contact homology, is out of reach of the methods presented here.

foundational difficulties, one can replace here, following, e.g., [9, 19], the linearized contact homology by the equivariant symplectic homology, which carries essentially the same information (see [8]), at the expense of proofs getting somewhat more involved; cf. Remarks 3.2.1 and 3.2.9, and [41] vs. [55] or [15] vs. [19].

Chapter 2

Conley Conjecture for Lagrangian Correspondences

2.1 Basic Definitions and Main Theorem

Given a symplectomorphism $M \xrightarrow{\phi} M$, its graph

$$\mathrm{Gr}(\phi) := \{(x, \phi(x)) \in M \times M \mid x \in M\}$$

is a Lagrangian submanifold in $M^- \times M := (M \times M, ((-\omega) \oplus \omega))$. Therefore, following [76], we shall consider a Lagrangian submanifold L in $M^- \times M$ as a relation on M, and denote it by $M \xrightarrow{L} M$; i.e. for $x, y \in M$, $x \overset{L}{\mapsto} y$ if and only if $(x, y) \in L$. Such a relation is called a *Lagrangian correspondence* on M. Notice that in general, a Lagrangian correspondence, considered as a relation, may fail to be the graph of a function. Nevertheless, one can still make sense of its periodic orbits as follows.

Definition 2.1.1. A *k-periodic orbit* of a Lagrangian correspondence $M \xrightarrow{L} M$ is a sequence of points $\{x_i\}_{i\in\mathbb{Z}_k} \subset M$ such that $(x_i, x_{i+1}) \in L$ for all $i \in \mathbb{Z}_k$. Such an orbit is said to be *simple* if $x_i \neq x_j$ for all $i, j \in \mathbb{Z}_k$, $i \neq j$.

Remark 2.1.2. The fixed points of the correspondence L constitute simply $L \cap \Delta_M$ where $\Delta_M := \{(x,x) \mid x \in M\}$ (notation-wise, if the context is clear, we usually omit the subscript M). Also, one can obtain the k-periodic orbits of a Lagrangian correspondence as the intersection of two Lagrangians as follows. Let $W = M^- \times M$ and $\cdot^T \colon W^k \to W^k$ be given by transposition of the last factor to the first. More precisely, the transposition is given by $(x_1, y_1, \dots, x_k, y_k)^T = (y_k, x_1, y_1, \dots, x_{k-1}, y_{k-1}, x_k)$, for $x_i, y_i \in M$ for $i = 1, \dots, k$. This map is anti-symplectic. Therefore it is symplectic as a map from W^k to $(W^k)^-$. Also let the twisted multi-diagonal $D^k := ((\Delta_M)^k)^T$ be the transpose of the k-fold product of the diagonal and $L^k = L \times \cdots \times L$ denote the k-fold product of the Lagrangian correspondence. It is rather straightforward to see that the k-periodic orbits $\{x_i\}_{i\in\mathbb{Z}_k}$ of L are in one-to-one correspondence with the intersection points $((x_i, x_{i+1}))_{i\in\mathbb{Z}_k}$ of L^k with D^k.

As pointed out in Section 1.1, we need further restrictions to the class of correspondences for which (CC) generalizes to (CC$^{\mathrm{LC}}$). Firstly, to draw parallel to the fact that the main object in (CC), i.e. φ, is a Hamiltonian diffeomorphism (rather than simply a symplectomorphism), we assume that the Lagrangian correspondence is Hamiltonian isotopic to the diagonal. To introduce the next assumption, we need to recall some definitions from [53]. The projection of a linear Lagrangian correspondence $\Lambda \subset V^- \times V$ to the first and the second component are called the *domain* and the *range* of Λ respectively, and their symplectic complements the *kernel* and the *halo* of Λ respectively.

Definition 2.1.3. A linear Lagrangian $\Lambda \subset V^{-} \times V$ is called *weakly non-degenerate* if its halo and domain (or equivalently its kernel and range) are transverse and the linear symplectic map has at least one eigenvalue not equal to 1 when the halo (or equivalently the kernel) is empty. A Lagrangian correspondence $M \xrightarrow{L} M$ is *weakly non-degenerate* if for every $x \in L \cap \Delta_M$, $\mathrm{T}_x L$ is weakly non-degenerate.

The dimension of the intersection of the halo and the image turns out to be an invariant, which can be used to classify Lagrangian correspondences, cf. [53]. To give a plethora of (linear) examples, consider a linear symplectic map $\phi \in \mathrm{Sp}(V_1)$ and a pair of transverse linear Lagrangians $\lambda_1, \lambda_2 \in \mathrm{Lag}(V_2)$, and, for $V = V_1 \oplus V_2$, the linear Lagrangian correspondence $L = \mathrm{Gr}(\phi) \oplus (\lambda_1 \times \lambda_2) \in \mathrm{Lag}(V^{-} \times V)$. Then the halo of L is λ_2, and its domain $V_1 \oplus (\lambda_1)$, which makes L weakly non-degenerate. Moreover, one can take $V_2 = \{0\}$ while ensuring the linear map to have at least one eigenvalue not equal to 1 for more examples. Following the same logic, we note that the graphs of weakly non-degenerate Hamiltonian diffeomorphisms are weakly non-degenerate Lagrangian correspondences since the kernel at every point is empty and the linear mapping is weakly non-degenerate. Therefore, this generalizes the weakly non-degeneracy in [62] to a suitable class of Lagrangian correspondences. Now, we are ready to state the main theorem.

Theorem 2.1.4. *Let $M \xrightarrow{L} M$ be a weakly non-degenerate Lagrangian correspondence of a closed symplectic Calabi–Yau manifold M ($c_1(\mathrm{T}M) = 0$) such that L is*

Hamiltonian isotopic to Δ_M. Then for any sufficiently large prime k, there exists a simple k-periodic orbit.

For the case $L = \mathrm{Gr}(\phi)$ where ϕ is a Hamiltonian diffeomorphism, we recover the Conley conjecture for weakly non-degenerate case (cf. [62]). As in the Hamiltonian diffeomorphism case, the requirement $c_1(TM) = 0$ is essential, e.g. for $M = S^2$ and ϕ the Hamiltonian diffeomorphism corresponding to an irrational rotation about an axis, there are no periodic points other than two fixed points. Likewise, the requirement that L is Hamiltonian isotopic to the diagonal is also essential since e.g. for $\mathbb{T}^2$, the Lagrangian correspondence $L = \mathrm{Gr}(\phi)$, where ϕ is a symplectic irrational translation, has no periodic orbits. Also as a counterexample in a more general setting, one can consider $L = l_1 \times l_2$ for Lagrangian submanifolds l_i of M such that $l_1 \cap l_2 = \emptyset$, in which case there are neither any fixed points nor any periodic points.

2.2 Linear Algebra of Lagrangian Correspondences

2.2.1 Decomposing Linear Lagrangian Correspondences

We shall investigate in depth linear Lagrangians $\Lambda \subset V^- \times V$. Recall that (V, ω) is symplectic and the symplectic structure Ω in $V^- \times V$ is given by $\Omega = (-\omega) \oplus \omega$. The space V^- called *the dual of* V, is simply V with the symplectic structure $-\omega$. Also let π_1, π_2 be projections onto the first and the second components respectively, and

$$\begin{aligned}
\operatorname{dom}(\Lambda) &= \{v \in V \mid \exists w \in V, (v,w) \in \Lambda\} = \pi_1(\Lambda) \\
\operatorname{ran}(\Lambda) &= \{w \in V \mid \exists v \in V, (v,w) \in \Lambda\} = \pi_2(\Lambda) \\
\ker(\Lambda) &= \{v \in V \mid (v,0) \in \Lambda\} \qquad\qquad\quad = \pi_1(\Lambda)^{\perp} \\
\operatorname{halo}(\Lambda) &= \{w \in V \mid (0,w) \in \Lambda\} \qquad\qquad\quad = \pi_2(\Lambda)^{\perp}
\end{aligned}$$

where the last one is called *the halo of* Λ and $C^{\perp}$ denotes the symplectic complement of C. The notation introduced above is standard for linear relations, see [67], whereas the name 'halo' is introduced in [53]. The final equalities for the kernel and the halo are exclusive for a linear Lagrangian Λ. Now, the following theorem establishes how much variation can be expected within a special class of linear Lagrangians.

Theorem 2.2.1. *For $\Lambda \subset V^{-} \times V$, assume that the halo and the domain are transverse. Then we have a symplectic decomposition $V = V_g \oplus V_p$ such that*

$$\Lambda = \operatorname{Gr}(\varphi) \oplus (\lambda_1 \times \lambda_2) \tag{2.1}$$

where ϕ is a symplectic linear map of V_g and λ_1, λ_2 are transverse Lagrangians in V_p.

Proof. Let $C_1 = \operatorname{dom}(\Lambda)$ and $C_2 = \operatorname{ran}(\Lambda)$. Notice that C_1, C_2 are coisotropic: if $v_1 \in C_1^{\perp}$, then $(v_1, 0) \in L^{\perp} = L$ since for every $(v,w) \in L$

$$\Omega\big((v_1,0),(v,w)\big) = -\omega(v_1,v) + \omega(0,w) = 0$$

where the last equality follows from the fact that $v_1 \in C_1^{\perp}$ and $v \in C_1$ by definition. Therefore $v_1 \in C_1$. Similarly, one can show that if $w_1 \in C_2^{\perp}$, then $(0,w_1) \in L$ and hence $w_1 \in C_2$. Now, consider

$$\tilde{\varphi} : C_1 \to C_2 \Big/ C_2^{\perp}$$

$$v \mapsto [w] \qquad \text{where } (v, w) \in L$$

This map is well-defined: If $(v, w), (v, w') \in L$ then $(0, w - w') \in L$ and for arbitrary $(\tilde{v}, \tilde{w}) \in L$, we have $0 = \Omega\big((0, w - w'), (\tilde{v}, \tilde{w})\big) = -\omega(0, \tilde{v}) + \omega(w - w', \tilde{w}) = \omega(w - w', \tilde{w})$, which implies that $w - w' \in C_2^{\perp}$ and hence $[w] = [w']$. Moreover,

$$\begin{aligned}
\ker \tilde{\varphi} &= \{v \in C_1 \mid [w] = 0\} \\
&= \{v \in C_1 \mid (v, w) \in L \text{ and } w \in C_2^{\perp}\} \\
&= \{v \in C_1 \mid \forall (\tilde{v}, \tilde{w}) \in L,\, \omega(v, \tilde{v}) = 0\} \\
&= C_1^{\perp}
\end{aligned}$$

where the penultimate equality follows from $0 = \Omega\big((\tilde{v}, \tilde{w}), (v, w)\big) = -\omega(\tilde{v}, v) + \omega(\tilde{w}, w) = -\omega(\tilde{v}, v)$. Therefore the quotient map $\varphi\colon C_1/C_1^{\perp} \to C_2/C_2^{\perp}$ is an isomorphism of symplectic subspaces of the same linear subspace. This is ensured by the assumption that $C_1 \cap C_2^{\perp} = \{0\}$. Calling this subspace V_g, we get the graph part in the decomposition.

To understand the product of Lagrangians appearing in the decomposition, assume without loss of generality that $V_g = \{0\}$. Then, C_1, C_2 are Lagrangian, and since $L \subseteq C_1 \times C_2$ by definition, $L = C_1 \times C_2$ follows. Moreover they are transverse since $\{0\} = C_1 \cap C_2^{\perp} = C_1 \cap C_2$. $\square$

Remark 2.2.2. The theorem is false without the transversality condition. Consider $\Lambda = \{(0, x, y, z, y, z, 0, t) \mid x, y, z, t \in \mathbb{R}\}$ as a Lagrangian linear relation $\mathbb{R}^4 \xrightarrow{\Lambda} \mathbb{R}^4$. The halo and the kernel intersect non-transversally and there is no decomposition since the only part that looks like a graph is from the second pair of the first 4-tuple to the first pair of the second 4-tuple. Moreover, following [53], the dimension of $\mathrm{halo}(\Lambda)\cap\mathrm{dom}(\Lambda)$ is an invariant of the linear relation encoding symplectic information, which present itself as decomposability in our case.

2.2.2 Mean Index for Paths of Lagrangian Correspondences

The aim of this section is to extend the definition of the mean index for paths of symplectic matrices to paths of Lagrangian correspondences from V to itself which start at the identity (which corresponds to the diagonal in $V^-\times V$) and which ends in $\mathcal{G} = \{\Lambda \in \mathrm{Lag}(V^- \times V) \bigm| \mathrm{halo}(\Lambda)\cap\mathrm{dom}(\Lambda) = \{0\}\}$. For completeness, we shall recall the definition of the mean index for paths of symplectic matrices here, following [62, 27].

For a path of symplectic matrices $\Phi\colon [0,1] \to \mathrm{Sp}(2n)$, the mean index $\hat{\mu}(\Phi)$ measures the total rotation angle of certain unit eigenvalues of $\Phi(t)$ and can be defined as follows. For $A \in \mathrm{Sp}(2)$, set $\rho(A) = e^{i\kappa}$ if A is conjugate to a counterclockwise rotation, $\rho(A) = e^{-i\kappa}$ if conjugate to a clockwise rotation, and $\rho(A) = \pm 1$ when A is hyperbolic with the sign of eigenvalues matching the sign of $\rho(A)$. Notice that ρ is a continuous conjugation invariant function, which restricts to the determinant over $U(2)$. For $A \in \mathrm{Sp}(2n)$ with distinct eigenvalues, let $\rho(A) = \prod_{j=1}^{k}\rho(A_j)$ where $A_j \in \mathrm{Sp}(2)$ such that A is conjugate to $\bigoplus_{j=1}^{k} A_j$. Again, ρ extends as a continuous conjugation invariant function that restricts to the determinant over $U(n)$. For the path Φ, set $\rho(\Phi(t)) = e^{i\kappa(t)}$ to get the total rotation of the preferred eigenvalues as $\hat{\mu}(\Phi) = (\kappa(1) - \kappa(0))/2$.

Throughout the exposition, let # denote the concatenation of two paths. More precisely, for $\gamma_1, \gamma_2 \colon [0,1] \to X$ such that $\gamma_1(1) = \gamma_2(0)$, the concatenation of γ_1 and γ_2 is given by

$$(\gamma_1 \# \gamma_2)(t) = \begin{cases} \gamma_1(2t), & t \in [0, \frac{1}{2}] \\ \gamma_2(2t-1), & t \in [\frac{1}{2}, 1] \end{cases}$$

Also, let γ^{-1} for a path $\gamma \colon [0,1] \to X$ denote the reversed path, i.e. $\gamma^{-1}(t) = \gamma(1-t)$.

Definition 2.2.3 (Mean Index for Paths of Linear Lagrangian Correspondences)**.**

Given

a path of linear Lagrangian correspondences $\tilde{\Lambda} \colon [0,1] \to \mathrm{Lag}(V^- \times V)$ where $\tilde{\Lambda}(0) = \Delta$ and $\Lambda := \tilde{\Lambda}(1) \in \mathcal{G}$, let $V_g, V_p, \phi, \lambda_i$ be as in Theorem 2.2.1 for Λ. Also let C be a loop based at Δ such that $\tilde{\Lambda}$ is homotopic to $C \# \operatorname{Gr}(\Phi(t) \oplus \Psi(t))_{t \in [0,1]}$ where

(a) $\Phi(t) \in \mathrm{Sp}(V_g)$ for $t \in [0,1]$ and $\Psi(t) \in \mathrm{Sp}(V_p)$ for $t \in [0,1)$

(b) $\Phi(0) = \mathbb{1}_{V_g}$ and $\Psi(0) = \mathbb{1}_{V_p}$

(c) $\Psi(t)$ is hyperbolic for $t \in (0,1)$

(d) $\Phi(1) = \phi$ and $\operatorname{Gr}(\Psi(t)) \xrightarrow{t \to 1} \lambda_1 \times \lambda_2$

Such paths always exists and the *mean index of the path* $\tilde{\Lambda}$ is

$$\hat{\mu}(\tilde{\Lambda}) = \mu(C) + \hat{\mu}(\Phi) \tag{2.2}$$

where $\mu(\cdot)$ denotes the Maslov index [60].

The existence of the path $\Psi(t)$ given in the definition (especially with the hyperbolicity requirement) might not be obvious at first. To that extend, given a pair of transverse Lagrangians λ_1, λ_2, let $\psi(t)$ be the hyperbolic map such that $\Psi(t)(v) = t^{-1}v_1 + tv_2$ where $v = v_1 + v_2$ with $v_i \in \lambda_i$. The map is well-defined since the transversality of the Lagrangian pair ensures that $V = \lambda_1 + \lambda_2$ and the decomposition $v = v_1 + v_2$ is unique. It is not hard to show that $\mathrm{Gr}(\Psi(t)) \xrightarrow{t\to\infty} \lambda_1 \times \lambda_2$ using projection operators (cf. Theorem 2.2.5). Moreover, the index is well-defined since any other choice of paths homotopic to $C\#\Phi$ would yield no change in both indices μ and $\hat{\mu}$ due to the fact that they are homotopically invariant.

Remark 2.2.4. The new mean index is homogeneous with respect to iterating via products, i.e $\hat{\mu}(\gamma^k) = k\hat{\mu}(\gamma)$ for a path γ as in Definition 2.2.3, where $\gamma^k\colon [0,1] \to \mathrm{Lag}((V^- \times V)^k)$ with $\gamma^k(t) = \gamma(t) \times \ldots \times \gamma(t)$. This is rather straightforward following the additive property of the Maslov index under products and the mean index for path of symplectic matrices. This also holds for the standard mean index with a crucial difference in the construction: In the identity $\hat{\mu}(\Phi^k) = k\hat{\mu}(\Phi)$, $\Phi^k\colon [0,1] \to \mathrm{Sp}(2n)$ is obtained by composition (cf. [62, Lemma 3.4]), not by taking products. The "composability" of Lagrangian correspondences as in [73, 49] is a topic in its own right and we emphasize that we do not impose any such condition on our correspondences.

It is rather straightforward to see that $\hat{\mu}(\Phi) = \hat{\mu}(\mathrm{Gr}(\Phi))$, therefore the new mean index agrees with the standard mean index for paths of linear symplectomorphisms. Therefore the next task is to prove the continuity of $\hat{\mu}$.

Theorem 2.2.5 (Continuity of the Mean Index)**.** *Let $\tilde{\Lambda}_n$ be a converging sequence of paths such that $\tilde{\Lambda}_n(1) \in \mathcal{G}$ for all n, and $\tilde{\Lambda}_n \xrightarrow{n\to\infty} \tilde{\Lambda}$ with $\tilde{\Lambda}(1) \in \mathcal{G}$. Then*

$\hat{\mu}(\Lambda_n) \xrightarrow{n\to\infty} \hat{\mu}(\tilde{\Lambda})$.

Proof. Since all the endpoints fall within $\mathcal{G}$, we can approximate each $\tilde{\Lambda}_n(1)$ by a sequence of graphs of paths of linear symplectomorphisms $\Psi_{n,m}(t)$ following the idea of paths in Definiton 2.2.3. Taking the diagonal sequence now, we obtain $\Psi_n(t)$ whose graphs as a path approximate $\tilde{\Lambda}$. Following the same definition, let C_n be a sequence of curves which approximates C such that the decomposition $\mathrm{Gr}(\Psi_n) = C_n \# \mathrm{Gr}(\tilde{\Psi}_n)$ approximates $\Lambda = C\#(\mathrm{Gr}(\Phi(t)) \oplus \lambda_1 \times \lambda_2)_{t\in[0,1]}$. Since the graph part converges to the standard mean index, we should investigate the behavior of linear symplectic maps whose graphs are arbitrarily close to a product of transversal Lagrangian pair.

Proposition 2.2.6. *Suppose that there is a sequence of symplectic matrices $\{A_n\}_{n\in\mathbb{N}}$ whose graphs $\{\mathrm{Gr}(A_n)\}_{n\in\mathbb{N}}$ converge to $\lambda_1 \times \lambda_2 \subset \mathrm{Lag}(\mathbb{R}^{4n})$, a product of two transverse Lagrangians $\lambda_1, \lambda_2 \in \mathrm{Lag}(\mathbb{R}^{2n})$. Then the eigenvalues of A_n are all eventually hyperbolic.*

Proof. A sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$ in the Lagrangian Grassmanian converges to Λ if the projection operators P_{Λ_n} converges to P_Λ. Hence, we should explore how the projections operators behave as the sequence converges. In order to compute the projection operator onto the graph of a matrix $\mathrm{Gr}(A)$, consider the matrix $B = \left[\begin{smallmatrix} I \\ A \end{smallmatrix}\right]$. The columns of B constitute a basis for $\mathrm{Gr}(A)$ and the projection operator can be computed as

$$P_A = B(B^TB)^{-1}B^T = \begin{bmatrix} I \\ A \end{bmatrix} \left(\begin{bmatrix} I & A^T \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix} \right)^{-1} \begin{bmatrix} I & A^T \end{bmatrix}$$

$$= \begin{bmatrix} I \\ A \end{bmatrix} (I + A^TA)^{-1} \begin{bmatrix} I & A^T \end{bmatrix}$$

Hence, for $S = (A^TA + I)^{-1}$, we have

$$P_A = \begin{bmatrix} S & SA^T \\ AS & ASA^T \end{bmatrix} \tag{2.3}$$

Notice that since S is symmetric, the projection matrix is symmetric, as well as all its blocks. We would like to investigate the behavior of eigenvalues of P_{A_n} as it converges to the block form

$$P_{\lambda_1 \times \lambda_2} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

where the blocks P_1, P_2 are projection operators to λ_1, λ_2 respectively. Thus we expect eigenspaces eventually converge to transversal pair of Lagrangians whereas the eigenvalues should either converge to 0 or 1. As it turns out, this behavior is due to a relationship between singular values of A and the eigenvalues of the blocks of P_A.

Definition 2.2.7. (σ, u, v) is *a singular triple* of A if $Au = \sigma v$ and $A^Tv = \sigma u$.

Notice that if (σ, u, v) is a singular triple of A, then σ is a singular value (in linear-algebra-sense) of A.

Lemma 2.2.8. *Let (σ, u, v) be a singular triple of A. Then*

(i) $((\sigma^2+1)^{-1}, u)$ *is an eigenvalue-eigenvector pair of S.*

(ii) $(\sigma^2(\sigma^2+1)^{-1}, v)$ *is an eigenvalue-eigenvector pair of ASA^T.*

(iii) $(\sigma(\sigma^2+1)^{-1}, u, v)$ *is a singular triple for AS.*

(iv) $(\sigma(\sigma^2+1)^{-1}, v, u)$ *is a singular triple for SA^T.*

Proof. By the definition of the singular triple, we have $A^TAu = \sigma A^Tv = \sigma^2 u$ and hence (i) follows. Similarly, $ASA^Tv = \sigma ASu = \sigma(\sigma^2+1)^{-1}Au = \sigma^2(\sigma^2+1)^{-1}v$ gives (ii); $ASu = (\sigma^2+1)^{-1}Au = \sigma(\sigma^2+1)^{-1}v$ and $SA^Tv = \sigma Su = \sigma(\sigma^2+1)^{-1}u$ gives (iii) and (iv). □

Now, assume that we have a sequence of matrices $\{A_n\}_n \in \mathbb{N}$ whose graphs converge to the product of two transverse Lagrangians λ_1, λ_2. Following Lemma 2.2.8 and the block-form of the projection matrix, AS and SA^T must converge to zero blocks and therefore all their singular values must converge to 0, i.e.

$$\lim_{n\to\infty} \frac{\sigma_{i,n}}{\sigma_{i,n}^2+1} = 0.$$

This is only possible for either $\sigma_{i,n} \xrightarrow{n\to\infty} 0$ or $\sigma_{i,n} \xrightarrow{n\to\infty} \infty$. Notice that in both cases, the eigenvalue-eigenvector pairs for S and ASA^T converge to 1 and 0 respectively, which forces the two spaces that we project on to be transversal. Since the singular values become eventually hyperbolic, the eigenvalues eventually become hyperbolic. This proves the proposition. □

Now, we are ready to complete the proof of the theorem. By Proposition 2.2.6, the linear maps that approximate $\lambda_1 \times \lambda_2$ eventually become hyperbolic, which forces the index $\hat{\mu}(\mathrm{Gr}(\Psi_n|_{V_p})) \xrightarrow{n\to\infty} 0$. □

Remark 2.2.9. The condition on λ_1, λ_2 being transversal in Proposition 2.2.6 (which is equivalent to $\Lambda \in \mathcal{G}$) is essential. As an example here, we construct a sequence of matrices whose graphs approach $\lambda \times \lambda \in \mathbb{R}^4$ while the rotation number can be arbitrary.

For fixed $\theta \in S^1 - \{\pm 1\}$, consider

$$A_n = \begin{bmatrix} \cos\theta + \left(2n^2 + \dfrac{1}{n^2}\right)\sin\theta & -\left(n^2 + \dfrac{1}{n^2}\right)\sin\theta \\ \left(4n^2 + \dfrac{1}{n^2}\right)\sin\theta & \cos\theta - \left(2n^2 + \dfrac{1}{n^2}\right)\sin\theta \end{bmatrix}$$

Notice that $x_n = \frac{1}{n}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $y_n = n\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ form a Darboux basis with

$$A_n x_n = \frac{1}{n}\begin{bmatrix} \cos\theta + n^2\sin\theta \\ \cos\theta + 2n^2\sin\theta \end{bmatrix} = (\cos\theta)x_n + (\sin\theta)y_n$$

$$A_n y_n = n\begin{bmatrix} \cos\theta - \dfrac{1}{n^2}\sin\theta \\ 2\cos\theta - \dfrac{1}{n^2}\sin\theta \end{bmatrix} = (-\sin\theta)x_n + (\cos\theta)y_n$$

So A_n is a fixed rotation on the aforementioned Darboux basis. To see where the graph converges, let

$$u_n = \frac{\csc\theta}{n}x_n \xrightarrow{n\to\infty} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad v_n = \frac{1}{n}y_n - \frac{\cot\theta}{n}x_n \xrightarrow{n\to\infty} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Clearly $\{u_n, v_n\}$ is a basis (not Darboux) for $n \in \mathbb{N}$ and we have

$$A_n u_n = \frac{\cot\theta}{n} x_n + \frac{1}{n} y_n \xrightarrow{n\to\infty} \begin{bmatrix} 1 \\ \\ 2 \end{bmatrix}$$

$$\begin{aligned} A_n v_n &= -\frac{\sin\theta}{n} x_n + \frac{\cos\theta}{n} y_n - \frac{\cos^2\theta}{n\sin\theta} x_n - \frac{\cos\theta}{n} y_n \\ &= -\frac{\csc\theta}{n} x_n \xrightarrow{n\to\infty} \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix} \end{aligned}$$

Therefore $\{(u_n, A_n u_n), (v_n, A_n v_n)\} \xrightarrow{n\to\infty} \{(0,0,1,2),(1,2,0,0)\}$, which means that the sequence of graphs $\mathrm{Gr}(A_n)$ converge to the product $\lambda \times \lambda$ with $\lambda = \mathrm{span}\{(1,2)\}$ while A_n is a rotation with a fixed angle on a varying basis.

2.2.3 Grading for Linear Lagrangian Correspondences and Index for Graded Linear Lagrangian Correspondences

This section focuses on developing grading and index to be used in intersection Floer homology.

Definition 2.2.10 (Grading of Linear Lagrangians)**.** A *grading* of $\Lambda \in \mathrm{Lag}(V^- \times V)$ is a lift $\tilde{\Lambda}$ of Λ to the universal cover $\widetilde{\mathrm{Lag}}(V^- \times V)$ realized as paths up-to-homotopy based at the diagonal, i.e. a path starting from the diagonal Δ and ending at Λ.

Definition 2.2.11 (Relative Maslov index [60, Section 3])**.** Given two graded Lagrangians $\tilde{\Lambda}_1, \tilde{\Lambda}_2 \in \widetilde{\mathrm{Lag}}(V^- \times V)$, let $\mathcal{C} = \{s \in [0,1] \mid \tilde{\Lambda}_1(s) \cap \tilde{\Lambda}_2(s) \neq \{0\}\}$ be the set of crossings, and, for $s \in \mathcal{C}$ and W_1, W_2 a pair of Lagrangian complements to

$\tilde{\Lambda}_1(s), \tilde{\Lambda}_2(s)$ respectively, let

$$\Gamma(\tilde{\Lambda}_1, \tilde{\Lambda}_2; s)v = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Omega(v, w_1(t) - w_2(t))$$

be *the crossing form* where $v \in \tilde{\Lambda}_1(s) \cap \tilde{\Lambda}_2(s)$ and $w_i(t) \in W_i$ such that $v + w_i(t) \in \tilde{\Lambda}_1(s+t)$ for small t. Assume that all crossing forms are non-degenerate (such a path is called *regular*). Now the relative Maslov index is defined as

$$\mu(\tilde{\Lambda}_1, \tilde{\Lambda}_2) = \frac{1}{2} \sum_{s \in \mathcal{C} \cap \{0,1\}} \operatorname{sign} \Gamma(\tilde{\Lambda}_1, \tilde{\Lambda}_2; s) + \sum_{s \in \mathcal{C} \cap (0,1)} \operatorname{sign} \Gamma(\tilde{\Lambda}_1, \tilde{\Lambda}_2; s)$$

Also, let

$$\mu_\delta(\tilde{\Lambda}_1, \tilde{\Lambda}_2) = \frac{1}{2} \sum_{s \in \mathcal{C} \cap \{0,1\}} \operatorname{sign} \Gamma(\tilde{\Lambda}_1, \tilde{\Lambda}_2; s) \text{ and } \mu_0(\tilde{\Lambda}_1, \tilde{\Lambda}_2) = \sum_{s \in \mathcal{C} \cap (0,1)} \operatorname{sign} \Gamma(\tilde{\Lambda}_1, \tilde{\Lambda}_2; s).$$

Due to Remark 2.1.2, since we realize k-periodic orbits as intersection of L^k with D^k, we shall need a grading for D^k in order to associate an intersection index to periodic points. Now we focus on how to define a canonical grading to the twisted multi-diagonal.

Following [73, Remark 3.0.5(b)], choose any Lagrangian $\lambda \in \operatorname{Lag}(V)$ and let

$$\gamma \colon = \overbrace{\left(\exp(tJ)\, \lambda^- \times \lambda\right)}^{\sigma_t}{}_{t \in [0, \frac{\pi}{2}]} \# \overbrace{\left(\{(tx + Jy, x + tJy) \colon x, y \in l\}\right)}^{\rho_t}{}_{t \in [0,1]}$$

This path connects $\lambda \times \lambda$ to Δ with vanishing Maslov index. Therefore taking a k-fold products and concatenating we get the canonical grading of the twisted multi-diagonal as

$$\widetilde{D^k} := [(\gamma^{-1} \times \cdots \times \gamma^{-1}) \# (\gamma \times \cdots \times \gamma)^T] \tag{2.4}$$

where γ^{-1} denotes the reversed path from Δ to $\lambda \times \lambda$ and $[\cdot]$ denotes the homotopy class of the path within.

Definition 2.2.12. The *iterated index* of a graded Lagrangian correspondence is given by $\mu(\tilde{\Lambda}, k) = \mu(\widetilde{\Lambda^k}, \widetilde{D^k})$ where the Λ^k is the k-fold product of the Lagrangian correspondence whose grading is derived from the product, and $\widetilde{D^k}$ as in (2.4).

The iterated index would be crucial in Section 2.5 as it is related directly to the degree in intersection Floer theory. More precisely, the degree [73, Definition 3.0.9] is given by

$$\mathrm{d}(\tilde{\Lambda}_1, \tilde{\Lambda}_2) = \frac{1}{2}\dim(\Lambda_1) + \mu(\tilde{\Lambda}_1, \tilde{\Lambda}_2) \tag{2.5}$$

or alternatively, following [73, Remark 3.0.10], $\mathrm{d}(\Delta, \tilde{\Lambda}) = \mu_0(\Delta, \tilde{\Lambda})$. Moreover, we have the following identity relating the iterated index to the aforementioned degree.

Proposition 2.2.13. $\mu(\tilde{\Lambda}, k) = \mathrm{d}(\widetilde{\Lambda^k}, \widetilde{D^k})$

Proof. It is straightforward to conclude that the representative in (2.4) is not regular at the end-point by computing the crossing form relative Δ. However, one can use the homotopy invariance of Maslov index to come up with another representative homotopic to (2.4) relative endpoints. The best way to understand both the homotopy class and the grading is through frames. Therefore, we start the proof by presenting frames for γ and introducing the new representative.

Since the grading of the path γ in (2.4) does not depend upon the choice of complex structure, fixing a basis $\{e_i\}_{i=1}^n$ for λ and completing it to a symplectic basis $\{e_i, f_i\}_{i=1}^n$, we can choose the compatible complex structure defined by $Je_i = f_i$ and assume without loss of generality that $\exp(tJ)$ becomes the standard rotation matrix in these coordinates. For $\epsilon_i(t) = \exp(tJ)e_i$, a suitable basis for σ_t is $\{(\epsilon_i(t), 0), (0, e_i)\}_{i=1}^n$. Notice that, $\epsilon_i(t) = (\cos t)e_i + (\sin t)f_i$ for $t \in [0, \frac{\pi}{2}]$.

Similarly, for ρ_t, a suitable basis is given by $\{(f_i, tf_i), (te_i, e_i)\}_{i=1}^n$.We can extend these bases as $(\sigma_t)^k = \operatorname{span}\{u_{i,j}(t), v_{i,j} \mid i = 1, \dots, n \text{ and } j = 1, \dots, k\}$ and $(\rho_t)^k = \operatorname{span}\{w_{i,j}(t), z_{i,j}(t) \mid i = 1, \dots, n \text{ and } j = 1, \dots, k\}$ with

$$\pi_m(u_{i,j}(t)) = (\delta_{jm}\epsilon_i(t), 0)$$

$$\pi_m(v_{i,j}) = (0, \delta_{jm}e_i)$$

$$\pi_m(w_{i,j}(t)) = \delta_{jm}(f_i, tf_i)$$

$$\pi_m(z_{i,j}(t)) = \delta_{jkm}(te_i, e_i)$$

where $\pi_m \colon (V^- \times V)^k \to V^- \times V$ denotes the projection onto the m^{th} component of the product, and δ_{jm} denotes the Kronecker delta. Since the grading for the transposed component is simply obtained by transposing the path γ^k, we can use the transposed bases for $(\Delta^k)^T$. Here, we modify the path slightly, specifically the second leg of the concatenation ρ_t^k to $\tilde{\rho}_t^k$ as follows. Let $\tau \colon [0,1] \to \mathbb{R}$ is a continuous function so that $\tau(0) = \tau(1) = 0$, $\tau(t) > 0$ for $t \in (0,1)$, and $\|\tau\|_\infty$ is small but non-zero, e.g. $\tau(t) = \varepsilon(t - t^2)$ for small $\varepsilon > 0$. Let $\tilde{\rho}_t^k(t) = \operatorname{span}\{\tilde{w}_{i,j}(t), \tilde{z}_{i,j}(t), i = 1, \dots, n,\ j = 1, \dots, k\}$ with

$$\pi_m(\tilde{w}_{i,j}(t)) = \delta_{mj}(-\sin\tau(t)e_i + \cos\tau(t)f_i, tf_i)$$

$$\pi_m(\tilde{z}_{i,j}(t)) = \delta_{mj}(te_i, \cos\tau(t)e_i + \sin\tau(t)f_i)$$

Thus $\tilde{\gamma}^k := (\sigma_t^k)_{t\in[0,\frac{\pi}{2}]}\#(\tilde{\rho}_t^k)_{t\in[0,1]}$ and $(\gamma^k)^T$ combined gives the modified representative for the grading $\widetilde{D^k} := [(\tilde{\gamma}^k)^{-1}\#(\tilde{\gamma}^k)^T]$. Now we are ready to prove that this representative (hence the grading associated to it) does not introduce any intersection index to the degree.

Lemma 2.2.14. *The index $\mu_0(\widetilde{D^k}, \Delta^k)$ vanishes and the crossing form relative Δ^k at the endpoint of $\widetilde{D^k}$ has signature 0.*

Proof. In the discussion below we shall denote the symplectic structure in $(V \times V^-)^k$ as $\Omega_k = \bigoplus_{j=1}^k (\omega) \oplus (-\omega)$. Moreover, since $V = \bigoplus_{i=1}^n \operatorname{span}\{e_i, f_i\}$ is a symplectic decomposition, without loss of generality, we shall assume that $n = 1$. Hence, to simplify the notation, we shall discard the index that runs through the basis of λ, e.g. $u_j = u_{i,j}$, $e = e_i$, etc. Moreover, using the concatenation property of the relative Maslov index [60, Theorem 2.3], $\mu(\widetilde{D^k}, \Delta^k) = \mu(\tilde{\gamma}^k, (\tilde{\gamma}^k)^T)$.

Assume that $s \in \left(0, \frac{\pi}{2}\right]$ is a regular crossing for the pair $(\sigma_t^k, (\sigma_t^k)^T)$. Let $a_j, b_j, \alpha_j, \beta_j$ be real coefficients for $j \in \mathbb{Z}/k\mathbb{Z}$, not all of them equal to 0, so that

$$\sum_{j'=1}^{k} a_{j'} u_{j'}(s) + b_{j'} v_{j'}(s) = \sum_{j'=1}^{k} \alpha_{j'} u_{j'}^T(s) + \beta_{j'} v_{j'}^T(s)$$

Using the projection π_j for $j \in \mathbb{Z}/k\mathbb{Z}$, we have

$$a_j \epsilon(s) = \beta_j e \text{ and } b_j e = \alpha_{j+1} \epsilon(s) \implies \left\{ \begin{matrix} a_j \cos s = \beta_j & b_j = \alpha_{j+1} \cos s \\ \\ a_j \sin s = 0 & 0 = \alpha_{j+1} \sin s \end{matrix} \right\}$$

Since $\sin s \neq 0$ on $\left(0, \frac{\pi}{2}\right]$, we should have $a_j = \alpha_j = 0$ for all $j \in \mathbb{Z}/k\mathbb{Z}$, and in turn, $\beta_j = b_j = 0$. Therefore, there are no intersections along the first part of the concatenation.

For the second part of the path, assume that $s \in (0, 1)$ is a regular crossing for the pair $(\tilde{\rho}_t^k, (\rho_t^k)^T)$. Using same symbols for the coefficient set and projecting via π_j, we get

$$(-a_j \sin\tau)e + (a_j \cos\tau)f + (sb_j)e = (s\alpha_j)f + (\beta_j)e$$

$$(sa_j)f + (b_j \cos\tau)e + (b_j \sin\tau)f = (\alpha_{j+1})f + (s\beta_{j+1})e$$

which yields

$$\begin{aligned} -a_j \sin\tau + sb_j &= \beta_j & a_j \cos\tau &= s\alpha_j \\ sa_j + b_j \sin\tau &= \alpha_{j+1} & b_j \cos\tau &= s\beta_{j+1} \end{aligned}$$

or equivalently

$$\begin{aligned} -\sin\tau a_j + sb_j &= \beta_j & a_j &= s\sec\tau\alpha_j \\ sa_j + \sin\tau b_j &= \alpha_{j+1} & b_j &= s\sec\tau\beta_{j+1} \end{aligned} \tag{2.6}$$

For a fixed parameter s, (2.6) is a linear system of equations in $c = (a_1, b_1, \dots, a_k, b_k)$ and $\theta = (\alpha_1, \beta_1, \dots, \alpha_k, \beta_k)$. Therefore, we shall introduce matrices to express them. Let

$$m_1 = \begin{bmatrix} 0 & s\sec\tau \\ s\sec\tau & 0 \end{bmatrix}, \quad m_2 = \begin{bmatrix} -\sin\tau & t \\ t & \sin\tau \end{bmatrix}$$

Also, let $M_1 = \Sigma^{-1}(\mathrm{diag}(m_1, \dots, m_1))$ and $M_2 = \Sigma(\mathrm{diag}(m_2, \dots, m_2))$, where diag denotes the block-diagonal matrix with blocks given in the argument, Σ denotes the transposition of the last row of any matrix to the first, and the matrices m_1, m_2 are repeated k times. In this setup, the equation set (2.6) can be compactly written as $c = M_1\theta$ and $\theta = M_2 c$. Thus $c = M_1\theta = M_1 M_2 c$ has a non-trivial solution if and only if $M := M_1 M_2$ has eigenvalue 1, i.e. $\det(M - I) = 0$. It is straightforward to show that for $\tilde{M} = \Sigma(M - I)$, we have

$$\tilde{M} = \left[\begin{array}{cccc|c} U & & & & V \\ V & U & & & \\ & V & \ddots & & \\ \hline & & & V & U \end{array}\right] := \left[\begin{array}{c|c} A & B \\ \hline C & U \end{array}\right]$$

where

$$U = \begin{bmatrix} -s\tan\tau & s^2\sec\tau \\ -1 & 0 \end{bmatrix} \text{ and } V = \begin{bmatrix} 0 & -1 \\ s^2\sec\tau & s\tan\tau \end{bmatrix}.$$

The determinant formula for the block matrices yield $\det(\tilde{M}) = \det(U)\det(A - BU^{-1}C)$ where

$$BU^{-1}C = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ VU^{-1}V & 0 & \cdots & 0 \end{bmatrix} \implies \det(A - BU^{-1}C) = \det(A) = (\det(U))^{k-1}$$

Here, we also used the fact that A is block-lower-diagonal with diagonal blocks U. Therefore $\det(\tilde{M}) = (\det(U))^k = (t^2\sec\tau)^k$. Since $\det(M)$ and $\det(\tilde{M})$ differ by a sign, and $\det(\tilde{M}) \neq 0$ by the choice of the function τ. Therefore there are no interior crossings, which completes the first part of the proof.

In order to compute the crossing form at the endpoint $t = 1$, we need to find complements to the Lagrangians. Thus, let $W = \operatorname{span}\{\zeta_j, \xi_j : j = 1, \ldots, k\}$ and $W' = \operatorname{span}\{\zeta_j^T, \xi_j^T : j = 1, \ldots, k\}$ where

$$\pi_m(\zeta_j) = \delta_{mj}(f, -f)$$

$$\pi_m(\xi_j) = \delta_{mj}(e, -e)$$

Notice that the intersection of the endpoints is

$$\tilde{\gamma}_1^k \cap (\gamma_1^k)^T = \{(v, v, \dots, v, v) \mid v \in V\} = \operatorname{span}\{(e, e, \dots, e, e), (f, f, \dots, f, f)\}$$

Our approach is as follows. Since the intersection is spanned by $v_1 = (e, e, \dots, e, e)$ and $v_2 = (f, f, \dots, f, f)$, the intersection form Γ can be realized as a bilinear form given by

$$\Gamma_{ij} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=1} \Omega_k(v_i, w_j(t) - \bar{w}_j(t))$$

Let $v_1 = (e, e, \dots, e, e)$, in order to compute the intersection form, we need to find $w_i(t) \in W$ such that $v_i + w_i(t) \in \tilde{\rho}_t^k$ and $v_i + \bar{w}_i(t) \in \tilde{(}\rho_t^k)^T$ for $i = 1, 2$.

To find $w_1(t) = \sum_{j=1}^n a_j(t)\zeta_j + b_j(t)\xi_j$, we need to compute the coefficients $a_j, b_j \in \mathbb{R}$ provided that $v_1 + w_1(t) \in \tilde{\rho}_t^k$. Notice that

$$\pi_m(v_1 + w_1(t)) = (1 + b_m)(e, 0) + a_m(f, 0) + (1 - b_m)(0, e) - a_m(0, f)$$

Therefore, in order for $v_1 + w_1(t) \in \tilde{\rho}_t^k$ to hold we need

$$\begin{aligned}\pi_m(v_1 + w_1(t)) &= \alpha_m(-\sin\tau(t)(e, 0) + \cos\tau(t)(f, 0) + t(0, f)) \\ &\quad + \beta_m(t(e, 0) + \cos\tau(t)(0, e) + \sin\tau(t)(0, f))\end{aligned}$$

Since this boils down to balancing coefficients α_m, β_m, assume that $\alpha_m = a_m \sec\tau$ and $\beta_m = (1 - b_m)\sec\tau$, which would balance the second and the third term in the expression of $\pi_m(v_1 + w_1(t))$. Thus we obtain following equations by balancing the first and the last terms.

$$\begin{aligned} 1 + b_m &= -a_m \tan\tau + (1 - b_m) t \sec\tau & & a_m(\tan\tau) + b_m(1 + t\sec\tau) = t\sec\tau - 1 \\ & & \Longrightarrow & \\ -a_m &= at\sec\tau + (1 - b_m)\tan\tau & & a_m(1 + t\sec\tau) - b_m(\tan\tau) = -\tan\tau \end{aligned}$$

$$\Longrightarrow \begin{bmatrix} \tan\tau & 1 + t\sec\tau \\ 1 + t\sec\tau & -\tan\tau \end{bmatrix} \begin{bmatrix} a_m \\ b_m \end{bmatrix} = \begin{bmatrix} t\sec\tau - 1 \\ -\tan\tau \end{bmatrix}$$

By inverting the matrix we can find the coefficients as

$$a_m = \frac{-2\tan\tau}{\tan^2\tau + (1 + t\sec^2\tau)^2} \quad \text{and} \quad b_m = \frac{\tan^2\tau + t^2\sec^2\tau - 1}{\tan^2\tau + (1 + t\sec^2\tau)^2}$$

and therefore

$$w_1(t) = \sum_{j=1}^{k} \frac{-2\tan\tau}{\tan^2\tau + (1 + t\sec^2\tau)^2}\zeta_j + \frac{\tan^2\tau + t^2\sec^2\tau - 1}{\tan^2\tau + (1 + t\sec^2\tau)^2}\xi_j$$

A similar computation for w_2 yields

$$w_2(t) = \sum_{j=1}^{k} \frac{1 - t^2\sec^2\tau - \tan^2\tau}{\tan^2\tau + (1 + t\sec^2\tau)^2}\zeta_j + \frac{-2t\sec\tau\tan\tau}{\tan^2\tau + (1 + t\sec^2\tau)^2}\xi_j$$

Therefore we have

$$\begin{aligned} \Omega_k(v_1, w_1) &= k\frac{2\tan\tau}{\tan^2\tau + (1 + t\sec\tau)^2} & \Omega_k(v_2, w_1) &= k\frac{\tan^2\tau + t^2\sec^2\tau - 1}{\tan^2\tau + (1 + t\sec\tau)^2} \\ \Omega_k(v_1, w_2) &= k\frac{\tan^2\tau + t^2\sec^2\tau - 1}{\tan^2\tau + (1 + t\sec\tau)^2} & \Omega_k(v_2, w_2) &= k\frac{-2t\sec\tau\tan\tau}{\tan^2\tau + (1 + t\sec\tau)^2} \end{aligned}$$

This completes the first terms needed for the computation of the intersection form.

Following the similar idea, to find $\bar{w}_1(t) = \sum_{j=1}^{n} \bar{a}_j(t)\zeta_j^T + \bar{b}_j(t)\xi_j^T$ provided that $v_1 + \bar{w}_1(t) \in (\tilde{\rho}_t^k)^T$, we compare the projections $\pi_m(v_1 + \bar{w}_1(t))$ to get

$$\begin{aligned} &(1 + \bar{b}_m)(e, 0) + \bar{a}_m(f, 0) + (1 - \bar{b}_m)(0, e) - \bar{a}_m(0, f) \\ &\qquad = \bar{\alpha}_m((f, 0) + t(0, f)) + \bar{\beta}_m(t(e, 0) + (0, e)) \end{aligned}$$

Firstly, comparing the coefficients of $(f, 0)$ and $(0, f)$, we realize that $\bar{a}_m$ must be zero. Secondly, comparing the coefficients of $(e, 0)$ and $(0, e)$ on both sides, we get

$$\frac{1+\bar{b}_m}{1-\bar{b}_m} = \frac{t\bar{\beta}_m}{\bar{\beta}_m} \implies \bar{b}_m = \frac{t-1}{t+1}$$

which yields $\bar{w}_1(t) = \frac{t-1}{t+1}\sum_{j=1}^{n} \xi_j^T$. As in the previous computation, one can similarly show that $\bar{w}_2(t) = \frac{1-t}{t+1}\sum_{j=1}^{n} \zeta_j^T$. Therefore we have

$$\begin{aligned} \Omega_k(v_1, \bar{w}_1) &= 0 & \Omega_k(v_2, \bar{w}_1) &= k\frac{t-1}{t+1} \\ \Omega_k(v_1, \bar{w}_2) &= k\frac{t-1}{t+1} & \Omega_k(v_2, \bar{w}_2) &= 0 \end{aligned}$$

Now, using the definition of the crossing form we compute

$$\Gamma_{11} = \frac{k}{2}\frac{\mathrm{d}\tau}{\mathrm{d}t}(1) \quad \Gamma_{21} = \Gamma_{12} = 0 \quad \Gamma_{22} = -\frac{k}{2}\frac{\mathrm{d}\tau}{\mathrm{d}t}(1) \tag{2.7}$$

which implies the signature of the crossing form is 0 at $t = 1$, hence it does not contribute to the index. □

Remark 2.2.15. Coming back to the problem with the initial representative to the grading in (2.4), if we let $\tau(t) = 0$ identically, then the last computation (2.7) shows that the crossing is not regular at the final point. Even though the crossing form at the final point does not enter into picture while computing $\mu_0(\widetilde{D^k}, \Delta^k)$, it is crucial while iterating $\mu_0(\tilde{L}, \Delta)$.

Corollary 2.2.16. $\mu(\tilde{\Lambda}, k) = \mu_0(\widetilde{\Lambda^k}, D^k)$ *where* D^k *denotes the constant path at* D^k *and* μ_0 *as in Definition 2.2.3.*

Proof. It follows from Lemma 2.2.14 and the homotopy invariance of μ that the only signatures in $\mu(\tilde{\Lambda}, k)$ that might arise should come from the endpoint D^k intersecting $\widetilde{\Lambda^k}$, which is precisely $\mu_0(\widetilde{\Lambda^k}, D^k)$. □

Now, to prove Proposition 2.2.13, notice that $\mathrm{d}(\tilde{\Lambda}, \widetilde{D^k}) = \mathrm{d}((\tilde{\gamma}^k)\#\tilde{\Lambda}, (\tilde{\gamma}^k)^T)$ by the concatenation property [60, Theorem 2.3]. Notice that both paths now originate at $(\lambda \times \lambda)^k$. Now, by [73, Remark 3.0.10], we can choose the crossing form at the beginning to be completely negative definite to cancel with the dimension term, and therefore get $\mathrm{d}(\tilde{\Lambda}, \widetilde{D^k}) = \mu_0(\tilde{\Lambda}, \widetilde{D^k})$. Here, we also use the fact that at the concatenation point (which is the $\tilde{\Lambda}(0)$) the intersection form has vanishing signature as proven in Lemma 2.2.14. Therefore $\mathrm{d}(\tilde{\Lambda}, \widetilde{D^k}) = \mu_0(\tilde{\Lambda}, D^k) = \mu(\tilde{\Lambda}, k)$. □

Remark 2.2.17. To reveal the relation of the iterated index with the Conley-Zehnder index (cf. [62, 60]), consider $\tilde{\Lambda}(t) = \mathrm{Gr}(\phi_t)$ for $\phi_t \in \mathrm{Sp}(V)$ such that $\phi(0) = \mathbb{1}_V$ and $\phi(1)$ weakly non-degenerate. Then we have $\mu(\tilde{\Lambda}, k) = \mu_0(\widetilde{\Lambda^k}, D^k)$ by Corollary 2.2.16. Moreover, notice that $(x_1, \phi(x_1), \ldots, x_k, \phi(x_k)) \in \tilde{\Lambda}^k \cap D^k$ is non-trivial if and only if $x_i = \phi(x_{i-1})$ for $i \in \mathbb{Z}/k\mathbb{Z}$, which is equivalent to $(x_1, \phi^k(x_1)) \in \Delta \cap \mathrm{Gr}(\phi^k)$. Therefore, $\mu(\tilde{\Lambda}, k) = \mu(\mathrm{Gr}(\phi^k), \Delta)$ which is nothing but the Conley-Zehnder index $\mu_{CZ}(\Phi^k)$ cf. [60, Remark 5.4].

2.3 Bounding The Gap Between Mean and Iterated Index

The aim of this section is to extend the index gap bound for weakly non-degenerate Hamiltionian diffeomorphisms (cf. [62]) to weakly non-degenerate Lagrangian correspondences. The grading of the Lagrangian correspondence plays a crucial role as it replaces the path that arises by the Hamiltionian diffeomorphism.

Theorem 2.3.1. *Given a graded weakly non-degenerate Lagrangian correspondence $\tilde{\Lambda}$, we have $|\mu(\tilde{\Lambda}, k) - k\hat{\mu}(\tilde{\Lambda})| < n$ for all k, where $2n = \dim(\Lambda)$.*

For $\tilde{\Lambda} = \mathrm{Gr}(\Phi)$ for $\Phi\colon [0,1] \to \mathrm{Sp}(V)$ a path of linear symplectomorphisms such that $\Phi(0) = \mathbb{1}_V$ and $\Phi(1)$ weakly non-degenerate, the theorem recovers [27, (CZ1)] with a strict inequality. We would like to emphasize again that the iteration in the Hamiltonian diffeomorphism case is composition whereas our iterations are taking products.

Proof. For given $\tilde{\Lambda}$, let $C, \Phi, \lambda_1, \lambda_2$ be as in Definition 2.2.3. Now the contribution of $\mu(C)$ to both $\mu(\tilde{\Lambda}, k)$ and $k\hat{\mu}(\tilde{\Lambda})$ is $k\mu(C)$ simply because the former is defined by products and the Maslov index is homogeneous with respect to products. Therefore, the loop does not contribute to the gap and we can analyze two separate cases using the decomposition in Theorem 2.2.1. If $\lambda_1 = \lambda_2 = \{0\}$, then Definition 2.1.3 forces ϕ to be weakly non-degenerate and the gap bound as in [27, (CZ1)] is satisfied with strict inequality with $2n = \dim V_g = \dim V$. Otherwise, a similar inequality $|\mu(\Phi^k) - k\mu(\Phi)| \leq n'$ is satisfied with possible equality, while $2n' = \dim V_g < \dim V = 2n$. Therefore, in both cases, we have $|\mu(\tilde{\Lambda}, k) - k\hat{\mu}(\tilde{\Lambda})| < n$. ☐

Corollary 2.3.2. *Given a graded weakly non-degenerate Lagrangian correspondence $\tilde{\Lambda}$, either $|\mu(\tilde{\Lambda}, k)| \xrightarrow{k\to\infty} \infty$ or $|\mu(\tilde{\Lambda}, k)|$ remains strictly bounded above by n for all k.*

Proof. Following Theorem 2.3.1, if the mean index $\hat{\mu}(\tilde{\Lambda})$ vanishes then the index remains strictly bounded. Otherwise, since $|k\hat{\mu}(\tilde{\Lambda})|$ increases unboundedly, the iterated index term $|\mu(\tilde{\Lambda}, k)|$ must increase unboundedly in order to keep up with the bound. $\square$

2.4 Lagrangian Floer Homology and Homology of Iterated Lagrangian Correspondences

The aim of this section is to establish homological results for Lagrangian correspondences. The main tool we will use is the Lagrangian Floer homology [16, 20], defined for a pair of Lagrangian submanifolds $L_0, L_1 \subset W$ whose chain complex comprises intersection points $L_0 \cap L_1$. In order to understand how this chain is graded, we need to introduce some structures required within the theory.

Let $\mathrm{Lag}(M)$ denote the fiberwise Lagrangian Grassmannian of M, i.e.

$$\mathrm{Lag}(M) = \{(x, \Lambda) \colon x \in M \text{ and } \Lambda \in \mathrm{Lag}(\mathrm{T}_x M)\}.$$

One can consider fiberwise universal covers of the Lagrangian Grassmannians and form a bundle covering $\mathrm{Lag}(M)$. Such a bundle exists provided that $c_1(TM) = 0$ [63, 44]. Moreover, the universal covers of $\mathrm{Lag}(M)$ is in one-to-one correspondence with $\widetilde{\mathrm{Sp}}(2n)$-structures on M, where $2n = \dim M$ (cf. [73, Section 3]). In our setting, we fix a universal cover and denote it by $\widetilde{\mathrm{Lag}}(M)$. Since this construction extends to products and duals, we obtain a universal cover $\widetilde{\mathrm{Lag}}(M^- \times M)$.

Definition 2.4.1. For $M \xrightarrow{L} M$, *a grading of* L is a lift of the map $L \to \mathrm{Lag}(M^- \times M)$ to $\sigma_L \colon L \to \widetilde{\mathrm{Lag}}(M^- \times M)$. A *grading of a symplectomorphism* $\varphi \colon M \to M$ is a lift of $\phi\colon\ \mathrm{Lag}(M) \to \mathrm{Lag}(M)$ to a bundle isomorphism $\tilde{\phi} \colon \widetilde{\mathrm{Lag}}(M) \to \widetilde{\mathrm{Lag}}(M)$.

Given two Lagrangians L_0, L_1 with respective gradings σ_0, σ_1, intersecting transversely at $x \in L_0 \cap L_1$, we can associate a degree

$$|x| = \mathrm{d}(\sigma_0(x), \sigma_1(x)) \tag{2.8}$$

with $\mathrm{d}(\cdot,\cdot)$ as in (2.5), which grades the chain complex of the Lagrangian intersection homology $HF(L_0, L_1)$

Following [73, Remark 3.0.7(c)], there is a canonical way of associating a grading to the diagonal, denoted by $\tilde{\Delta} \in \widetilde{\mathrm{Lag}}(M^- \times M)$, fixed throughout the discussion. Now we would like to introduce grading for Lagrangian correspondences $M \xrightarrow{L} M$ Hamiltonian isotopic to the diagonal. Therefore, we shall assume that $L = \varphi(\Delta)$ with a Hamiltonian isotopy φ_t such that $\varphi = \varphi_1$ and $\varphi_0 = \mathbb{1}_{M^- \times M}$. Thus, by [73, Remark 3.0.7(b)], the map φ has a grading. More precisely, the Hamiltonian diffeomorphism φ induces first an isomorphism of bundles $\phi\colon \mathrm{Lag}(M^- \times M) \longrightarrow \mathrm{Lag}(M^- \times M)$, which can be lifted to an isomorphism between the universal covers $\tilde{\phi}\colon \widetilde{\mathrm{Lag}}(M^- \times M) \longrightarrow \widetilde{\mathrm{Lag}}(M^- \times M)$ using the lift of the identity map, which is derived from the grading of the diagonal $\Delta = \mathrm{Gr}(\mathbb{1}_M)$ as in [73, Remark 3.0.7(c)], together with the Hamiltonian isotopy φ_t. Now following [73, Remark 3.0.7(f)], the Lagrangian correspondence L considered as an image $\varphi(\Delta)$ gets a grading σ_L, which is the composition of the grading of the diagonal with the grading of the Hamiltonian diffeomorphism φ.

We are interested in a single Lagrangian correspondence $M \xrightarrow{L} M$ and its iterations. The points of interest are intersection of $L^k = L \times \cdots \times L$ with D^k as pointed out in Remark 2.1.2. Therefore, the intersection Floer homology that we are interested in is $HF(L,k) := HF(L^k, D^k)$. We would like to understand how the generators of the chain complex of $HF(L,k)$, which are nothing but k-periodic orbits, behave under iteration since the result we are seeking states the existence of infinitely many simple orbits. Therefore the key step towards the result is through computing the cohomology for arbitrary k.

Theorem 2.4.2. *Let L be a weakly non-degenerate Lagrangian correspondence, Hamiltonian isotopic to the diagonal. Then*

$$HF_*(L,k) \cong H_{*+n}(M)$$

for any k.

Proof. By definition $HF_*(L,k) = HF_*(L^k, D^k)$, and, due to the isotopy assumption, L^k is Hamiltonian isotopic to Δ^k. Sinee the Lagrangian Floer homology is invariant under Hamiltonian isotopies, we have $HF_*(L^k, D^k) \cong HF_*(\Delta^k, D^k)$. Now, since Δ^k and D^k intersect cleanly along the little diagonal $\mathcal{D} = \{(x, x, \ldots, x, x) \colon x \in M\} \subset (M^- \times M)^k$, by [58, Corollary 3.4.13] $HF_*(L^k, D^k) \cong H_{*+n}(M)$. Therefore, $HF_*(L,k) \cong H_{*+n}(M)$. □

2.5 Proof of Theorem 2.1.4

Assume that we have a weakly non-degenerate Lagrangian correspondence, Hamiltonian isotopic to the diagonal, together with a grading for the diagonal $\tilde{\Delta}$ as a section of the universal cover bundle $\widetilde{\mathrm{Lag}}(M^- \times M)$. Moreover, assume for a contradiction that the correspondence has finitely many simple periodic orbits whose periods are $k_1, \ldots, k_N$. Let $K = \mathrm{lcm}(k_1, \ldots, k_N)$, the least common multiple of the principle. Let $\mathcal{L} = L^K$ and consider $HF(\mathcal{L}, k)$. Note that $\mathcal{L}$ has only fixed points due to the assumption that there are only finitely many periodic orbits. Now, since the iterated index of all fixed points of $\mathcal{L}$ needs to satisfy Corollary 2.3.2, the highest degree of the homology cannot have any generators, due to the fact that the index n could never be achieved. Therefore, $\mathcal{L}$ must have at least one simple periodic orbit, which means that the original correspondence L must have another simple periodic orbit, which is a contradiction. This completes the proof of Theorem 2.1.4 as promised.

Chapter 3

Resonance Relations for Closed Reeb Orbits

3.1 Iterated index

In this section, we establish a few elementary results concerning the count, with multiplicity and signs, of periodic orbits of smooth maps. Although we feel that these results must be known in some form, we are not aware of any reference; see however [10, 12], and also [42, Chapter 3] and [66, Section I.4], for related arguments. Throughout the section, all maps are assumed to be at least C^1-smooth unless explicitly stated otherwise.

3.1.1 Iterated index of a map

As a model situation, consider a C^1-smooth map $F\colon M \to M$, where M is a closed manifold. We are interested in an algebraic count of periodic orbits of F.

To state our main results, let us first review some standard definitions and facts.

Denote by $\mathrm{Fix}(F)$ the set of the fixed points of F. Recall that x is a κ-periodic point of F if $F^\kappa(x) = x$, i.e., $x \in \mathrm{Fix}(F^\kappa)$, and that τ is the *minimal period* of x if τ is the smallest positive integer such that $F^\tau(x) = x$. The κ-periodic orbit containing x is the collection $\mathcal{O} = \{x, F(x), \ldots, F^{\kappa-1}(x)\}$ of not necessarily distinct points (naturally parametrized by $\mathbb{Z}/\kappa\mathbb{Z}$). The minimal period is then the length (i.e., the cardinality) of the orbit or, more precisely, of its image in M. Note that necessarily $\tau|\kappa$.

The *index* $I(F, x)$ of an isolated fixed point x of F is the degree of the map $S^{n-1} \to S^{n-1}$ given, in a local chart containing x, by

$$z \mapsto \frac{z - F(z)}{\|z - F(z)\|}$$

where z belongs to a small sphere S^{n-1} centered at x. It is not hard to see that all points in a κ-periodic orbit $\mathcal{O}$ have the same index as fixed points of F^κ. Thus, setting $I(F^\kappa, \mathcal{O}) := I(F^\kappa, x)$ for any $x \in \mathcal{O}$, we have the index assigned to a κ-periodic orbit.

Furthermore, recall that a fixed point x of F is said to be *non-degenerate* if 1 is not an eigenvalue of the linearization $DF_x \colon T_xM \to T_xM$ and that F is called non-degenerate if all its fixed points are non-degenerate. To proceed, let us first assume that all periodic points of F are non-degenerate, i.e., all iterations F^κ are non-degenerate. (As follows from (a part of) the Kupka–Smale theorem, this is a C^∞-generic condition; see, e.g., [1, 66].) Then the index $I(F^\kappa, x)$ is equal to $(-1)^m$, where m is the number of real eigenvalues of DF^κ_x in the range $(1, \infty)$.

Definition 3.1.1. Let x be a periodic point of F with minimal period τ. We say that x is *even (odd)* is the number of real eigenvalues of $DF^{\tau}_x : T_xM \to T_xM$ in the interval $(-\infty, -1)$ is even (odd). A κ-periodic point x with minimal period τ is said to be *bad* if it is an even iteration of an odd point, i.e., x is odd and the ratio κ/τ is even. Otherwise, x is said to be *good*. A κ-periodic orbit $\mathcal{O}$ is bad (good) if one, or equivalently all, periodic points in $\mathcal{O}$ are bad (good).

Alternatively, the difference between even and odd (or good and bad) periodic points can be seen as follows. Let x be a periodic point with minimal period τ. We can also view x as a κ-periodic point for any positive integer κ divisible by τ. Then $I(F^{\kappa}, x) = I(F^{\tau}, x)$ when x is even and $I(F^{\kappa}, x) = (-1)^{1+\kappa/\tau} I(F^{\tau}, x)$ when x is odd. Finally, x, viewed as a κ-periodic point, is good or bad depending on whether $I(F^{\kappa}, x) = I(F^{\tau}, x)$ or not. Since all periodic points in a periodic orbit have the same index, this definition extends to periodic orbits.

The terminology we use here is borrowed from the theory of contact homology (see, e.g., [6, 13]). In dynamics, odd periodic orbits are sometimes also referred to, at least for flows, as Möbius orbits (see [10]) or flip orbits. Furthermore, note that the above discussion relies heavily on the fact that no root of unity is an eigenvalue of DF^{k}_x due to the non-degeneracy assumption.

As is well known, the Lefschetz number $I(F) := \sum_{x \in \mathrm{Fix}(F)} I(F, x)$ is a homotopy invariant of F; see, e.g., [18, 42]. (In particular, $I(F)$ can be extended "by continuity" to all, not necessarily non-degenerate, maps F.) This, of course, applies to F^{κ} as well, and hence $I(F^{\kappa}) = \sum_{x \in \mathrm{Fix}(F^{\kappa})} I(F^{\kappa}, x)$, the number of periodic points counted with signs, is also homotopy invariant. However, the number of κ-periodic orbits, taken again with signs, is not homotopy invariant, i.e., $\sum_{\mathcal{O}} I(F^\kappa, \mathcal{O})$, where the summation extends to all (not necessarily simple) κ-periodic orbits, can vary under a deformation of F. (An example is, for instance, the second iteration of the period doubling bifurcation map in dimension one, starting with an attracting fixed point with an eigenvalue in $(-1, 0)$ and ending with an odd repelling fixed point and an (even) attracting orbit of period two, with the sum for $\kappa = 2$ changing from 1 before the bifurcation to 0 afterwards; cf. [77].) However, this sum is very close to being homotopy invariant and it becomes such once the summation is restricted to good orbits only. Namely, assuming as above that all periodic points of F are non-degenerate, set

$$I_\kappa(F) := \sum_{\text{good } \mathcal{O}} I(F^\kappa, \mathcal{O}),$$

where the sum is now taken over all good κ-periodic orbits $\mathcal{O}$ of F, not necessarily with minimal period κ. We call $I_\kappa(F)$ the *iterated index* of F. In the example of a period doubling bifurcation mentioned above, we have $I_2(F) = 1$ before and after the bifurcation. We emphasize that $I_\kappa(F)$ depends in general not only on F^κ, but separately on κ and F.

Let φ be the Euler function, i.e., $\varphi(\kappa)$ is the number of positive integers which are smaller than κ and relatively prime with κ. By definition, $\varphi(1) = 1$.

Theorem 3.1.2. *Assume, as above, that F^κ is non-degenerate. Then*

$$I_\kappa(F) = \frac{1}{\kappa} \sum_{d|\kappa} \varphi(\kappa/d) I(F^d). \tag{3.1}$$

Since the right hand side of (3.1) is obviously homotopy invariant, so is $I_\kappa(F)$. Furthermore, it extends by continuity to all F, not necessarily meeting the non-degeneracy requirement. Namely, let $\tilde{F}$ be a small perturbation of F such that all κ-periodic points of $\tilde{F}$ are non-degenerate. Then $I_\kappa(F) := I_\kappa(\tilde{F})$ is well defined, i.e., independent of the perturbation $\tilde{F}$.

Example. Assume that F is homotopic to the identity. Then, so is F^d for all d, and, by Euler's formula (see (3.2) below), we have $I(F^\kappa) = I(F) = I_\kappa(F)$ for all $\kappa \geq 1$.

Proof of Theorem 3.1.2. A κ-periodic orbit $\mathcal{O}$ with minimal period $\tau = |\mathcal{O}|$ contributes to $I_\kappa(F)$ only when $\tau|\kappa$, and to the individual terms on the right hand side of (3.1) when $\tau|d$. Set $\kappa = \tau\kappa'$ and $d = \tau d'$.

When $\mathcal{O}$ is good, its contribution to $I_\kappa(F)$ is equal to $I(F^\kappa, \mathcal{O}) = I(F^\tau, \mathcal{O})$. On the other hand, its contribution to the right hand side of (3.1) is

$$\frac{1}{\kappa}\sum_{\tau|d|\kappa}\varphi(\kappa/d)I(F^d,\mathcal{O})\tau = \frac{1}{\kappa'}\sum_{d'|\kappa'}\varphi(\kappa'/d')I(F^\tau,\mathcal{O}) = I(F^\tau,\mathcal{O}),$$

where we use Euler's formula

$$\sum_{l|r}\varphi(l) = r; \tag{3.2}$$

see, e.g., [33, Theorem 63].

Assume next that $\mathcal{O}$ is bad. Then $\kappa' = \kappa/\tau$ is even and $\mathcal{O}$, viewed as a τ-periodic orbit, is odd, and $I(F^d, \mathcal{O}) = (-1)^{1+d/\tau} I(F^\tau, \mathcal{O})$. Thus the contribution of $\mathcal{O}$ to $I_\kappa(F)$ is zero and its contribution to the right hand side is

$$\frac{1}{\kappa}\sum_{\tau|d|\kappa}\varphi(\kappa/d)I(F^d,\mathcal{O})\tau = -\frac{1}{\kappa'}\sum_{d'|\kappa'}\varphi(\kappa'/d')(-1)^{d'}I(F^\tau,\mathcal{O}) = 0,$$

where we now use the fact that

$$\sum_{l|r}(-1)^l\varphi(r/l) = 0 \tag{3.3}$$

for any positive even integer r.

Finally, (3.3) can be proved, for instance, as follows. Set $r = 2^m c$, where c is odd and $m \geq 1$ since r is even. Then, regrouping the terms on the left hand side of (3.3), we have

$$\sum_{l|r}(-1)^l\varphi(r/l) = \sum_{a|c} P(a),$$

where

$$P(a) = (-1)^{2^m a}\varphi(c/a) + (-1)^{2^{m-1}a}\varphi(2c/a) + \ldots + (-1)^a\varphi(2^m c/a),$$

and it suffices to show that $P(a) = 0$. Set $b = c/a$ and recall that $\varphi(2^j b) = 2^{j-1}\varphi(b)$ since φ is multiplicative and b is odd; [33]. When $m = 1$, we clearly have $P(a) = \varphi(b) - \varphi(b) = 0$. Likewise, as is easy to see,

$$P(a) = \varphi(b) + \varphi(b) + \ldots + 2^{m-2}\varphi(b) - 2^{m-1}\varphi(b) = 0$$

when $m \geq 2$. This completes the proof of the theorem. $\square$

Remark 3.1.3. The fact that the iterated index is homotopy invariant enables one to extend the definition of $I_\kappa(F)$ to continuous maps $F\colon M \to M$, when M is still a smooth manifold, by setting $I_\kappa(F) = I_\kappa(\tilde{F})$ where $\tilde{F}$ is a smooth approximation of F. Clearly, Theorem 3.1.2 still holds in this case.

3.1.2 Iterated index of a germ

Consider now a germ of a C^1-smooth map F at a fixed point x, which is assumed to be isolated for all iterations F^κ but not necessarily non-degenerate. Thus the index $I(F^\kappa, x)$ is defined and homotopy invariant as long as x remains *uniformly isolated*: $I(F_s^\kappa, x) = const$ when F_s varies smoothly or continuously with parameter s and there exists a neighborhood U of x such that x is the only fixed point of F_s^κ in U for all s.

For a given κ, consider a sufficiently small perturbation $\tilde{F}$ such that all κ-periodic points of $\tilde{F}$ are non-degenerate and set

$$I_\kappa(F, x) := \sum_{\text{good } \mathcal{O}} I(\tilde{F}^\kappa, \mathcal{O}),$$

where the sum is again taken over all good κ-periodic orbits $\mathcal{O}$ of $\tilde{F}$. For instance, when F and all its iterations are non-degenerate, we have $I_\kappa(F, x) = I(F, x)$ when x is even or when x is odd and κ is odd, and $I_\kappa(F, x) = 0$ otherwise. In what follows, when the point x is clear from the context, we will use the notation $I(F)$ and $I_\kappa(F)$.

Theorem 3.1.4. *We have*

$$I_\kappa(F) = \frac{1}{\kappa} \sum_{d|\kappa} \varphi(\kappa/d) I(F^d). \tag{3.4}$$

We omit the proof of this theorem; for it is word-for-word identical to the proof of Theorem 3.1.2. An immediate consequence is

Corollary 3.1.5. *The iterated index $I_\kappa(F, x)$ is well defined and homotopy invariant as long as x is a uniformly isolated fixed point of F_s^κ.*

Remark 3.1.6. As in the case of maps of smooth manifolds (see Remark 3.1.3), the definition of the iterated index $I_\kappa(F, x)$ extends to continuous maps F and Theorem 3.1.4 remains valid. In what follows, however, the requirement that F is at least C^1 becomes essential; cf. [64].

Assuming that x is isolated for all iterations F^κ, consider the sequence of indices $\iota_\kappa = I(F^\kappa)$. This sequence is bounded (see [64]) and in fact periodic; see [10] and also, e.g., [42] and references therein. (It is essential here that F is at least C^1-smooth.) Moreover, there exists a finite collection of positive integers $\mathcal{N}$ with the following properties:

(i) $1 \in \mathcal{N}$;

(ii) for any two elements q and q' in $\mathcal{N}$, the least common multiple $\mathrm{lcm}(q, q')$ is also an element of $\mathcal{N}$;

(iii) for any κ, we have $\iota_\kappa = \iota_{q(\kappa)}$, where $q(\kappa)$ is the largest element in $\mathcal{N}$ dividing κ.

Thus the index sequence ι_κ is $q_{\max}$-periodic, where $q_{\max} = \max \mathcal{N}$, and takes values in the set $\{\iota_q \mid q \in \mathcal{N}\}$. The collection $\mathcal{N} = \mathcal{N}(F)$ is generated in the obvious sense by the degrees of the roots of unity occurring among the eigenvalues of the linearization DF_x and, in addition, $2 \in \mathcal{N}$ when x is odd. (For instance, if all eigenvalues of DF_x are equal to 1, we have $\mathcal{N} = \{1\}$ and the sequence ι_κ is constant; when none of the eigenvalues is a root of unity, i.e., x is non-degenerate for all iterations, we have either $\mathcal{N} = \{1\}$ or $\mathcal{N} = \{1, 2\}$ depending on whether x is even or odd.)

Condition (iii) relating the sequence and the set $\mathcal{N}$ satisfying (i) and (ii) is particularly important, and we say that a sequence a_κ is *subordinated* to $\mathcal{N}$ when (iii) holds: for any κ, we have $a_\kappa = a_{q(\kappa)}$, where $q(\kappa)$ is the largest element in $\mathcal{N}$ dividing κ. (Note that (iii) is an easy consequence of the fact, implicitly contained in the proof of the Shub–Sullivan theorem [64], that $I(F^\kappa) = I(F)$ wherever κ is relatively prime with all $q \in \mathcal{N}$; cf. [10, 26].)

Theorem 3.1.7. *The sequence $I_\kappa(F)$ is subordinated to $\mathcal{N}$.*

In particular, it follows that $I_\kappa(F)$ is also periodic with period $q_{\max}$.

Proof. The result is a formal consequence of Theorem 3.1.4. Namely, let a_κ be any sequence subordinated to a finite set $\mathcal{N}$ satisfying (i) and (ii), and let b_κ be obtained from a_κ via (3.4):

$$b_\kappa = \frac{1}{\kappa} \sum_{d|\kappa} \varphi(\kappa/d) a_d.$$

Then we claim that b_κ is also subordinated to $\mathcal{N}$.

At this point the fact that the original sequences are integer-valued becomes inessential, and it is more convenient to assume that the sequences in question are real. We can view the transform $\{a_\kappa\} \mapsto \{b_\kappa\}$ as a map Φ from the vector space $\mathbb{R}^{\mathcal{N}}$ of all real sequences subordinated to $\mathcal{N}$ to the vector space of all sequences. Our goal is to show that Φ actually sends $\mathbb{R}^{\mathcal{N}}$ to itself.

Consider the basis $\delta(q)$, $q \in \mathcal{N}$, of $\mathbb{R}^{\mathcal{N}}$, where $\delta(q)_\kappa = 1$ when $q|\kappa$ and $\delta(q)_\kappa = 0$ otherwise. Then, as we will prove shortly, Φ is diagonal in this basis with entries $1/q$, i.e., $\Phi\big(\delta(q)\big) = \delta(q)/q$. In particular, $\Phi\colon \mathbb{R}^{\mathcal{N}} \to \mathbb{R}^{\mathcal{N}}$ and the theorem follows.

By definition,

$$\Phi\big(\delta(q)\big)_\kappa = \frac{1}{\kappa}\sum_{d|\kappa}\varphi(\kappa/d)\delta(q)_d.$$

The only non-zero terms on the right hand side are those with $q|d$. Thus, when κ is not divisible by q, the right hand side is zero and hence $\Phi\big(\delta(q)\big)_\kappa = 0$. When $q|\kappa$, set $\kappa = q\kappa'$ and $d = qd'$ as in the proof of Theorem 3.1.2. Then

$$\Phi\big(\delta(q)\big)_\kappa = \frac{1}{q\kappa'}\sum_{d'|\kappa'}\varphi(\kappa'/d') = \frac{1}{q},$$

where we again used (3.2). This completes the proof of the theorem. □

Remark 3.1.8 (Integrality). In addition to being subordinated to $\mathcal{N}$, the sequence $\iota_\kappa = I(F^\kappa)$ is known to satisfy some further conditions. Namely, [10, Theorem 2.2] asserts that, in the notation from the proof of Theorem 3.1.7, the sequence ι_κ is an integral linear combination of the sequences $q\delta(q)$, where $q \in \mathcal{N}$. (Furthermore, the coefficients must meet certain "sign-reversing" constraints if x is odd.) Moreover, it follows that the mean index (the φ-index in the terminology of [10]) is an integer:

$$\bar{\iota} = \lim_{N\to\infty}\frac{1}{N}\sum_{\kappa=1}^{N}\iota_\kappa = \frac{1}{q_{\max}}\sum_{\kappa=1}^{q_{\max}}\iota_\kappa \in \mathbb{Z};$$

see [10, Corollary 2.3].

Arguing as in the proof of Theorem 3.1.7, one can derive [10, Theorem 2.2] from the fact that the right hand side of (3.4) is an integer by the definition of $I_\kappa(F)$, and conversely [10, Theorem 2.2] implies that the right hand side of (3.4) is an integer since $\Phi(q\delta(q)) = \delta(q)$.

Example. Assume that $2n = 2$ and F is elliptic with eigenvalues $e^{\pm 2\pi i\alpha}$, where α is rational: $\alpha = p/q$ with p and q relatively prime and $q \geq 2$. Then it follows from [10, Theorem 2.2] or our Theorem 3.1.7 that $I(F^\kappa) = 1 - rq$ when $q|\kappa$ and $I(F^\kappa) = 1$ otherwise. Thus $I_\kappa(F) = 1 - r$ when $q|\kappa$ and $I_\kappa(F) = 1$ otherwise. In addition, when F is area preserving, $I(F^\kappa) \leq 1$ (see [56, 65]), and hence $r \geq 0$ in this case. There are no other restrictions on the index sequence when $2n = 2$: all values of r and q do occur. For instance, in the area preserving case, one can take as F the composition of the rotation $e^{2\pi i/q}$ with the flow of the "monkey saddle" Hamiltonian $\Re(z^{qr})$. One immediate consequence of this result, otherwise entirely non-obvious, is that $I(F^\kappa) \neq 0$ for all κ, when $I(F) \neq 0$. (Remarkably, under some additional assumptions, these facts remain true for homeomorphisms; see [47, 48] and references therein.)

3.2 Mean Euler characteristic

3.2.1 Notation and conventions

Let (M^{2n-1}, ξ) be a closed contact manifold strongly fillable by an exact symplectically aspherical manifold. In other words, we require (M, ξ) to admit a contact form α such that there exists an exact symplectic manifold $(W, \omega = d\alpha_W)$ with $M = \partial W$ (with orientations) and $\alpha_W|_M = \alpha$ and $c_1(TW) = 0$. Then the linearized contact homology $\mathrm{HC}_*(M, \xi)$ is defined and independent of α; see, e.g., [6, 13]. Although $\mathrm{HC}_*(M, \xi)$ depends in general on the filling (W, ω), we suppress this dependence in the notation. Moreover, when M is clear from the context, we will simply write $\mathrm{HC}_*(\xi)$.

When α is non-degenerate, $\mathrm{HC}_*(\xi)$ is the homology of a complex $\mathrm{CC}_*(\alpha)$

generated over $\mathbb{Q}$ by the good closed, not necessarily simple, Reeb orbits x of α, where an orbit is said to be good/bad depending on whether the corresponding fixed point of the Poincaré return map is good/bad. The complex is graded by the Conley–Zehnder index up to a shift of degree by $n-3$, i.e., $|x| = \mu_{CZ}(x) + n - 3$. The exact nature of the differential on $\mathrm{CC}_*(\alpha)$ is inessential for our considerations.

The complex $\mathrm{CC}_*(\alpha)$ further breaks down into a direct sum of sub-complexes $\mathrm{CC}_*(\alpha; \gamma)$ generated by the closed Reeb orbits in the free homotopy class γ of loops in W. When $\gamma \neq 0$, fixing grading (or, equivalently, a way to evaluate the Conley–Zehnder index of x) requires fixing an extra structure. A convenient choice of such an extra structure in our setting is a non-vanishing section $\mathfrak{s}$, taken up to homotopy, of the square of the complex determinant line bundle $\left(\bigwedge_{\mathbb{C}} TW\right)^{\otimes 2}$; see [15]. (The section $\mathfrak{s}$ exists since $c_1(TW) = 0$.) With this choice, for every closed Reeb orbit x, its Conley–Zehnder index $\mu_{CZ}(x)$ and the mean index $\Delta(x)$ are well-defined. Namely, the indices of x are evaluated using a (unitary) trivialization of x^*TW such that the square of its top complex wedge is $\mathfrak{s}|_x$. Such a trivialization is unique up to homotopy. Moreover, the mean index is homogeneous with respect to the iteration, i.e., $\Delta(x^\kappa) = \kappa\Delta(x)$. We refer the reader to [15] for a very detailed discussion of the mean index in this context and for further references, and also to [52, 62]. Here we only mention that the mean index $\Delta(x)$ measures the total rotation of certain eigenvalues on the unit circle of the linearized Poincaré return map of the Reeb flow at x. In general, the grading does depend on the choice of $\mathfrak{s}$.

Finally, let us also fix a collection Γ of free homotopy classes in W closed under iterations, i.e., such that $\gamma^\kappa \in \Gamma$ whenever $\gamma \in \Gamma$. (For instance, we can have $\Gamma = \{0\}$ or Γ can be the entire collection of free homotopy classes.) In what follows, we focus on the homology $\bigoplus_\Gamma \mathrm{HC}_*(\xi, \gamma)$. We again suppress the dependence of the homology on $\mathfrak{s}$ and Γ (and on W) in the notation and $\mathrm{HC}_*(\xi)$.

Furthermore, assume that

(CH) there are two integers l_+ and l_- such that the space $\mathrm{HC}_l(\xi)$ is finite-dimensional for $l \geq l_+$ and $l \leq l_-$.

In all examples considered here the contact homology is finite dimensional in all degrees and this condition is automatically met. Set

$$\chi^\pm(\xi) = \lim_{N\to\infty} \frac{1}{N} \sum_{l=l_\pm}^{N} (-1)^l \dim \mathrm{HC}_{\pm l}(\xi), \tag{3.5}$$

provided that the limits exist. We call $\chi^\pm(\xi)$ the *positive/negative mean Euler characteristic (MEC)* of ξ. (Invariants of this type for contact manifolds are originally introduced and studied in [69, Section 11.1.3]; see also [15, 29, 59].) Note that $\chi^\pm(\xi)$ depends of course on Γ and also, when $\Gamma \neq \{0\}$, on $\mathfrak{s}$. (However, when, say, M is simply connected, it is not hard to see that $\chi^\pm(\xi)$ is independent of the filling under some natural additional conditions on ξ; cf. [9, 45, 57].)

Assume now that the dimensions of the contact homology spaces remain bounded as $l \to \pm\infty$:

$$\dim_{\mathbb{Q}} \mathrm{HC}_l(\xi) \leq \mathit{const} \text{ when } |l| \geq l_\pm. \tag{3.6}$$

Then, although the limit in (3.5) need not exist, one can still define $\chi^\pm(\xi)$ as, e.g., following [9],

$$\chi^{\pm}(\xi) = \frac{1}{2}\left[\limsup_{N\to\infty}\frac{1}{N}\sum_{l=l_{\pm}}^{N}(-1)^l \dim \mathrm{HC}_{\pm l}(\xi) + \liminf_{N\to\infty}\frac{1}{N}\sum_{l=l_{\pm}}^{N}(-1)^l \dim \mathrm{HC}_{\pm l}(\xi)\right].$$

Finally, let us also point out that the machinery of contact homology used in this section is yet to be fully put on a rigorous basis and the foundations of the theory is still a work in progress; see [39, 40].

Remark 3.2.1. One can also use the cylindrical contact homology, when it exists, to define the MEC of a contact manifold (M, ξ). The construction is similar to the one for the linearized contact homology with obvious modifications. For instance, in the cylindrical case, $\mathfrak{s}$ is a non-vanishing section of the line bundle $\left(\bigwedge_{\mathbb{C}} \xi\right)^{\otimes 2}$ and the collection Γ is formed by free homotopy classes of loops in M. Of course, the two definitions give the same result when the cylindrical and contact homology groups are equal or when the dimension of the contact complex is bounded as a function of the degree and the MEC can be evaluated using the complex rather than the homology. This is the case, for example, in the setting of Theorem 3.2.2.

Alternatively, the MEC can be defined using the positive equivariant symplectic homology (see [9, 19]), resulting in an invariant (somewhat hypothetically) equal to the one obtained using the linearized contact homology; [8]. Note that this approach bypasses the foundational difficulties related to contact homology but usually results in somewhat more involved proofs and calculations; cf. Remark 3.2.9. Finally, variants of the MEC exist and have been used for "classical" homology theories; see, e.g., [51, 59, 72].

3.2.2 Examples

In this section we briefly review, omitting the proofs, some examples where the MEC is not difficult to determine.

Example (The Standard Sphere). For the standard contact structure ξ_0 on S^{2n-1}, the contact homology is one-dimensional in every even degree starting with $2n-2$; see, e.g., [6]. Thus, in this case, $\chi^+ = 1/2$ and $\chi^- = 0$. Alternatively, one can use here the Morse–Bott approach as in Example 3.2.2; see [15].

Example (The Ustilovsky Spheres). In [68], Ustilovsky considers a family of contact structures ξ_p on S^{2n-1} for odd n and positive $p \equiv \pm 1 \mod 8$. For a fixed n, the contact structures ξ_p fall within a finite number of homotopy classes, including the class of the standard structure ξ_0. The contact homology $\mathrm{HC}_*(\xi_p)$ is computed in [68], and it is not hard to see that in this case

$$\chi^+(\xi_p) = \frac{1}{2}\left(\frac{p(n-1)+1}{p(n-2)+2}\right) \tag{3.7}$$

and $\chi^-(\xi_p) = 0$; see [45, 69] and also [15, 29] for a different approach. The right-hand side of (3.7) is a strictly increasing function of $p > 0$. Hence, χ^+ distinguishes the structures ξ_p with $p > 0$. Note also that $\chi^+(\xi_p) > \chi^+(\xi_0) = 1/2$ when $p > 1$ and $\chi^+(\xi_1) = 1/2$. In particular, χ^+ distinguishes ξ_p with $p > 1$ from the standard structure ξ_0.

Example (Pre-quantization Circle Bundles). Let $\pi\colon M^{2n-1} \to B$ be a prequantization circle bundle over a closed strictly monotone symplectic manifold (B, ω). In other words, M is an S^1-bundle over B with $c_1 = [\omega]$, i.e., we have $\pi^*\omega = d\alpha$, where

α is a connection form on M and we use a suitable identification of the Lie algebra of S^1 and $\mathbb{R}$; see, e.g., [31, Appendix A]. As is well known, α is a contact form. Assume first for the sake of simplicity that the fiber x of π is contractible. Then, for $\Gamma = \{0\}$, we have $\chi^-(\xi) = 0$ and

$$\chi^+(\xi) = \frac{\chi(B)}{2\,\langle c_1(TB), u\rangle},$$

where $u \in \pi_2(B)$ is the image of a disk bounded by x in M and $\chi(B)$ is the ordinary Euler characteristic of the base B. (Here we are using the cylindrical contact homology of (M, ξ) rather than the linearized contact homology because the natural filling of M by the disk bundle is not exact or even symplectically aspherical.) This is an easy consequence, for instance, of the Morse–Bott version of (3.9) and of the fact that, essentially by definition, the denominator is $\Delta(x)$; see [15, Example 8.2]. Furthermore, since x is contractible, $\langle \omega, u\rangle = 1$ and B is monotone, $\langle c_1(TB), u\rangle$ is the minimal Chern number N of B; cf. [5, p. 100]. To summarize, we have

$$\chi^+(\xi) = \frac{\chi(B)}{2N}. \tag{3.8}$$

If x is not contractible, the above calculation still holds for $\Gamma = \{0\}$. More generally, when the order r of the class $[x]$ in $\pi_1(M)$ is finite, for the collection Γ generated by $[x]$ and any $\mathfrak{s}$, we have $\chi^+(\xi) = r\chi(B)/(2N)$. When B is negative monotone, the roles of χ^+ and χ^- are interchanged. (See [9] for far reaching generalizations of this calculation.)

Example (The Unit Cotangent Bundle of S^n). Let (M, ξ) be the unit cotangent bundle ST^*S^n with the standard contact structure ξ. A Riemannian metric on

S^n gives rise to a contact form α on M and, for the standard round metric, α is a prequantization contact form as in Example 3.2.2, where $B = \mathrm{Grass}^+(2, n+1)$ is the real "oriented" Grassmannian. (The closed geodesics are the oriented great circles on S^n, i.e., the intersections of S^n with the oriented 2-planes in $\mathbb{R}^{n+1}$.) It is not hard to see that $\chi(B) = 2\lfloor (n+1)/2 \rfloor$. Furthermore, since the minimal Chern number N of a (simply connected) hypersurface of degree d in $\mathbb{CP}^{m+1}$ is $m + 2 - d$ and since B is a complex quadric in $\mathbb{CP}^n$, we have $N = n - 1$; cf. [46, p. 88] and [61, Example 4.27 and Exercise 6.20]. Thus $\chi^-(\xi) = 0$ and $\chi^+(\xi) = 1/2 + 1/(n-1)$ if $n \geq 3$ is odd and $\chi^+(\xi) = 1/2 + 1/2(n-1)$ when $n \geq 2$ is even. When $n = 2$ we have $\chi^+(\xi) = 1/2$ for $\Gamma = \{0\}$ and $\chi^+(\xi) = 1$ for $\Gamma = \pi_1(M) = \mathbb{Z}_2$. Alternatively, $\chi^\pm(ST^*S^n)$ can be calculated directly via contact homology; see [69, 45]. Note that here one can use either the cylindrical contact homology or the linearized contact homology with the filling W of M by the unit ball bundle in T^*S^n. (The only modification that is needed in the latter case is that when $n = 2$, the fiber x is contractible in W and $\chi^+(\xi) = 1$ for $\Gamma = \{0\}$.)

We refer the reader to [9, 15, 69, 45] for further examples.

3.2.3 Local formula for the MEC

Let now x be a (simple) closed orbit of the Reeb flow of α, which we assume to be isolated for all iterations, i.e., all iterated orbits x^κ are isolated. Note that these orbits are not required to be non-degenerate. The Poincaré return map F of x is a germ of a smooth map with an isolated fixed point which we also denote by x. Clearly, the Poincaré return map of the iterated orbit x^κ is just F^κ.

Since the fixed point x of F is isolated for all iterations, the iterated indices $I_\kappa(F,x)$ are defined for all κ and the sequence $I_\kappa(F,x)$ is periodic with period $q_{\max}$; see Section 3.1.2.

Set the *mean iterated index* of x to be

$$\sigma(x) = \frac{1}{q_{\max}} \sum_{\kappa=1}^{q_{\max}} I_\kappa(F,x) = \lim_{N\to\infty} \frac{1}{N} \sum_{\kappa=1}^{N} I_\kappa(F,x).$$

This is a diffeomorphism (and even homeomorphism) invariant of the flow near the orbit x. When x and all its iterations are non-degenerate, we have $\sigma(x) = I(F,x)$ when x is even and $\sigma(x) = I(F,x)/2$ when x is odd. (A simple closed Reeb orbit is said to be even/odd depending on whether the fixed point x is even/odd for F.)

Recall that (M^{2n-1},ξ) is a closed contact manifold strongly fillable by an exact symplectically aspherical manifold and that a filling (W,ω), the homotopy class of the section $\mathfrak{s}$ and the collection Γ are fixed, and hence, in particular, we have the graded space $\mathrm{HC}_*(\xi)$ and the mean indices of the orbits unambiguously defined.

Theorem 3.2.2. *Assume that the Reeb flow of α on (M^{2n-1},ξ) has only finitely many simple periodic orbits $x_1,\ldots,x_r$, not necessarily non-degenerate, in the collection Γ. Then the conditions (CH) and* (3.6) *are satisfied with $l_+ = 2n-3$ and $l_- = 3$. Furthermore, the limit in* (3.5) *exists for both the positive and negative MEC of (M,ξ), and*

$$\chi^\pm(\xi) = \sum\nolimits^\pm \frac{\sigma(x_i)}{\Delta(x_i)}, \tag{3.9}$$

where $\sum^\pm$ stands for the sum taken over the orbits x_i with positive/negative mean index $\Delta(x_i)$.

This result generalizes the non-degenerate case of (3.9) proved in [29] and inspired by [70] where such a formula was established for (S^{2n-1}, ξ_0). A variant of (3.9) for convex hypersurfaces in $\mathbb{R}^{2n}$ is proved in [72]. The general form of (3.9) given here is literally identical to a MEC formula obtained in [41] (see [41, Theorem 1.5]), and probably to the one established in [51] for (S^{2n-1}, ξ_0), as can be seen by comparing the proofs. The main difference lies in the definitions of the numerators: in [41], $\sigma(x_i)$ is defined as the MEC of the local contact homology of x_i, which is a contact invariant, while here $\sigma(x_i)$ is defined purely topologically as a diffeomorphism and even homeomorphism invariant of the Poincaré return map. (Likewise, in [51], the numerators are defined via certain local homology associated with x_i.) Finally note that the Morse–Bott version of (3.9) and its variant for the so-called asymptotically finite contact manifolds are proved in [15]. (The Morse–Bott version for the geodesic flows is originally established in [59].)

Remark 3.2.3. The condition of Theorem 3.2.2 that ω is exact on W can be slightly relaxed. Namely, when $\Gamma = \{0\}$, it suffices to assume that $\omega|_{\pi_2(W)} = 0$. In general, when $\Gamma \neq \{0\}$, it is enough to require ω to be atoroidal; cf. [28]. However, the assumption that $c_1(TW) = 0$ or at least that $c_1(TW)|_{\pi_2(W)} = 0$ appears to be essential.

3.2.4 Preliminaries and the proof

Our goal in this section is to prove Theorem 3.2.2. To this end, we need first to recall several definitions and results concerning filtered and local contact homology following mainly [41] and [28]. Throughout the section, we continue to work in the setting of Section 3.2.1. In particular, the contact manifold (M, ξ) is fixed as are the background structures used in the definition of the contact homology (the filling W, the section $\mathfrak{s}$ and the collection Γ). A contact form, α, is always assumed to support ξ, i.e., $\ker \alpha = \xi$.

The proof of Theorem 3.2.2 has non-trivial overlaps with the proof of its counterpart in [41] – after all, the difference between the two results lies mainly in the interpretation of (3.9) – although our argument is rather more concise. Moreover, our proof is not entirely self contained and depends on [41] at two essential points. One is the construction of the local contact homology and the other is one of its properties. This is (LC3) stated in Section 3.2.4; see also [28, Theorem 3 and Corollary 1]. We further elaborate on the relation between the two theorems and show how, once some preliminary work is done, our Theorem 3.2.6 can be derived from [41, Theorem 1.5] in Remark 3.2.6.

Filtered contact homology

Let us first assume that the contact form α is non-degenerate. The complex $\mathrm{CC}_*(\alpha)$ is filtered by the action

$$\mathcal{A}_\alpha(x) = \int_x \alpha,$$

i.e., its subspace $\mathrm{CC}_*^a(\alpha)$, where $a \in \mathbb{R}$, generated by the orbits x with $\mathcal{A}_\alpha(x) \leq a$ is a subcomplex. We refer to the homology $\mathrm{HC}_*^a(\alpha)$ of this complex as the *filtered contact homology*; see, e.g., [28, 41]. Note that, in contrast with $\mathrm{HC}_*(\xi)$, these spaces depend on α. Whenever $a \leq b$, we have a natural map

$$\mathrm{HC}_*^a(\alpha) \to \mathrm{HC}_*^b(\alpha)$$

induced by the inclusions of the complexes and, since homology commutes with direct limits,

$$\varinjlim_{a\to\infty} \mathrm{HC}_*^a(\alpha) = \mathrm{HC}_*(\xi). \tag{3.10}$$

This definition extends to degenerate forms by continuity. Below we will discuss this definition in detail; for the construction is essential for what follows and it involves certain non-obvious (to us) nuances. Let us start, however, by recalling some standard facts and definitions.

The action spectrum $\mathcal{S}(\alpha)$ of α is the collection of action values $A_\alpha(x)$, where x ranges through the set of closed Reeb orbits of α (in the class Γ). This is a closed zero-measure subset of $\mathbb{R}$.

For any two contact forms α and α', giving rise to the same contact structure ξ, write $\alpha' > \alpha$ when $\alpha'/\alpha > 1$, i.e., $\alpha' = f\alpha$ with $f > 1$. This is clearly a partial order, and, when both forms are non-degenerate, we have a homomorphism $\mathrm{HC}_*^a(\alpha') \to \mathrm{HC}_*^a(\alpha)$ induced by the natural cobordism in the symplectization of ξ between α and α'.

Furthermore, we will need the following invariance property of filtered contact homology, stated in a slightly different form in [28, Proposition 5]: *Let α_s, $s \in [0, 1]$, be a family of contact forms such that α_0 and α_1 are non-degenerate and $a \notin \mathcal{S}(\alpha_s)$ for all s. Then the contact homology spaces $\mathrm{HC}_*^a(\alpha_0)$ and $\mathrm{HC}_*^a(\alpha_1)$ are isomorphic and the isomorphism is independent of the family α_s as long as $a \notin \mathcal{S}(\alpha_s)$.*

Furthermore, assume that the family α_s is decreasing, i.e., $\alpha_{s'} > \alpha_s$ when $s' < s$. Then the natural map $\mathrm{HC}_*^a(\alpha_0) \to \mathrm{HC}_*^a(\alpha_1)$ is an isomorphism. This proposition can be proved in exactly the same way as its Floer homological counterparts; see, e.g., [3, 22, 71].

Let now α be a possibly degenerate contact form and $a \notin \mathcal{S}(\alpha)$. Set

$$\mathrm{HC}_*^a(\alpha) := \mathrm{HC}_*^a(\tilde{\alpha}),$$

where $\tilde{\alpha}$ is a non-degenerate C^∞-small perturbation of α, i.e., $\tilde{\alpha}$ is non-degenerate and C^∞-close to α. By the invariance property of the filtered contact homology, for any two non-degenerate perturbations sufficiently close to α the homology groups on the right hand side are canonically isomorphic, and hence the left hand side is well defined. Alternatively, we could have set

$$\mathrm{HC}_*^a(\alpha) := \varinjlim_{\tilde{\alpha}>\alpha} \mathrm{HC}_*^a(\tilde{\alpha}),$$

where the limit is taken over all non-degenerate forms $\tilde{\alpha} > \alpha$. These two definitions are obviously equivalent. In contrast with $\mathrm{HC}_*(\xi)$, the graded vector spaces $\mathrm{HC}_*^a(\alpha)$ are automatically finite dimensional: $\sum_l \dim \mathrm{HC}_l^a(\alpha) < \infty$.

With this definition, we have a well-defined map $\mathrm{HC}_*^a(\alpha') \to \mathrm{HC}_*^a(\alpha)$ for any two, not necessarily non-degenerate, forms $\alpha' > \alpha$, and hence the non-degeneracy requirement on α_0 and α_1 in the invariance property can be dropped. Now, however, once α is not assumed to be non-degenerate, (3.10) requires a proof. Implicitly, this result is already contained in [41].

Lemma 3.2.4. *For any, not necessarily non-degenerate form α,* (3.10) *holds as $a \to \infty$ through $\mathbb{R} \setminus \mathcal{S}(\alpha)$.*

Proof. Fix a sequence $a_i \to \infty$ with $a_i \notin \mathcal{S}(\alpha)$ and consider a decreasing sequence of non-degenerate forms α_j C^∞-converging to α from above, i.e., we have

$$\alpha_1 > \alpha_2 > \cdots > \alpha.$$

These sequences give rise to the maps

$$\ldots \to \mathrm{HC}_*^{a_i}(\alpha_j) \to \mathrm{HC}_*^{a_i}(\alpha_{j+1}) \to \ldots . \tag{3.11}$$

In addition, of course, we also have the homomorphisms

$$\ldots \to \mathrm{HC}_*^{a_i}(\alpha_j) \to \mathrm{HC}_*^{a_{i+1}}(\alpha_j) \to \ldots \tag{3.12}$$

commuting with the maps (3.11). After passing to the limit as $i \to \infty$, the maps (3.11) induce the identity map on $\mathrm{HC}_*(\xi)$.

Set

$$L = \varinjlim_{a_i \to \infty} \mathrm{HC}_*^{a_i}(\alpha_i)$$

with respect to the "diagonal" maps

$$\delta_i \colon \mathrm{HC}_*^{a_i}(\alpha_i) \to \mathrm{HC}_*^{a_{i+1}}(\alpha_{i+1}).$$

To prove the lemma, it suffices to show that $L \cong \mathrm{HC}_*(\xi)$.

Let $u \in \mathrm{HC}_*(\xi)$. By (3.10), there exists $i(u)$ such that u is the image of $u_{i(u)1} \in \mathrm{HC}_*^{a_{i(u)}}(\alpha_1)$. Applying the maps (3.11) and (3.12) to $u_{i(u)1}$, we obtain a double sequence $u_{ij} \in \mathrm{HC}_*^{a_i}(\alpha_j)$ where $i \geq i(u)$. Set $v_i = u_{ii}$, $i \geq i(u)$. Clearly, $\delta_i(v_i) = v_{i+1}$, and thus the sequence v_i gives rise to an element $v \in L$. As a result, we have constructed a homomorphism

$$\Phi \colon \mathrm{HC}_*(\xi) \to L, \quad \Phi(u) = v.$$

Conversely, consider a sequence $v_i \in \mathrm{HC}_*^{a_i}(\alpha_i)$ with $i \geq i(v)$ such that $\delta_i(v_i) = v_{i+1}$. The image u of v_i in $\mathrm{HC}_*(\xi)$ is independent of choice of v_i, and we have a map

$$\Psi\colon L \to \mathrm{HC}_*(\xi), \quad \Psi(v) = u.$$

Essentially by the definition, $\Psi\Phi = \mathbb{1}$. $\square$

Finally let $\mathrm{HC}_*^{(a,\,b)}(\alpha)$, where $a < b$, be the homology of the quotient complex $\mathrm{CC}_*^b(\alpha)/\,\mathrm{CC}_*^a(\alpha)$, provided that α is non-degenerate. When $a < b$ are outside $\mathcal{S}(\alpha)$ this definition again extends to all α by continuity. In any event, we obviously have the exact sequence

$$\ldots \to \mathrm{HC}_*^a(\alpha) \to \mathrm{HC}_*^b(\alpha) \to \mathrm{HC}_*^{(a,\,b)}(\alpha) \to \ldots.$$

Local contact homology

Let x be an isolated, but not necessarily simple or non-degenerate, closed Reeb orbit of α. Consider a small non-degenerate perturbation $\tilde{\alpha}$ of α. Under this perturbation x splits into, in general, several non-degenerate orbits $\tilde{x}_1, \ldots, \tilde{x}_r$ with nearly the same period (i.e., action) and mean index as x. The vector space $\mathrm{CC}_*(\tilde{\alpha}, x)$ generated over $\mathbb{Q}$ by the good orbits in this collection can be equipped with a differential. The resulting homology, denoted by $\mathrm{HC}_*(x)$, is independent of the perturbation $\tilde{\alpha}$; see [41]. We call $\mathrm{HC}_*(x)$ the *local contact homology* of x. These complexes and the homology spaces are graded just as the ordinary contact homology, i.e., $|\tilde{x}_i| = \mu_{CZ}(\tilde{x}_i) + n - 3$.

Example. Assume that x is non-degenerate. Then $\mathrm{HC}_*(x)$ is equal to $\mathbb{Q}$ and concentrated in degree $|x|$ when x is good and $\mathrm{HC}_*(x) = 0$ when x is bad.

Example. Assume that x is simple. Then $\mathrm{HC}_*(x)$ is isomorphic up to a shift of degree to the local Floer homology $\mathrm{HF}_*(F)$ of the Poincaré return map F of x. See [28, 41] for the proof of this fact, which can also be established by repeating word-for-word the proof of [14, Proposition 4.30], and, e.g., [23, 26, 55] for a detailed discussion of local Floer homology.

Example. When x is not simple, the relation between the local contact homology and the Floer homology is more involved. Namely, $\mathrm{HC}_{*+n-3}(y^\kappa) = \mathrm{HF}_*^{\mathbb{Z}_\kappa}(F^\kappa)$, where F is the Poincaré return map of a simple orbit y and the right hand side stands for the $\mathbb{Z}_\kappa$-equivariant local Floer homology; [28, Theorem 3]. The proof of this theorem depends on the machinery of multivalued perturbations and is only outlined in [28]. Although we find this relation illuminating, the present paper does not rely on this result.

The proof of Theorem 3.2.2 makes use of several properties of local contact homology, which we now recall following [28, 41]:

(LC1) $\mathrm{HC}_*(x)$ is finite dimensional and supported in the range of degrees $[\Delta(x) - 2, \Delta(x) + 2n - 4]$, i.e., $\mathrm{HC}_*(x)$ vanishes when $*$ is outside this range.

(LC2) Assume that $x = y^\kappa$, where y is a simple closed Reeb orbit, and let F be the Poincaré return map of y. Then, in the notation of Section 3.1.2,

$$\sum (-1)^l \dim \mathrm{HC}_l(x) = I_\kappa(F, y).$$

(LC3) For all iterations κ of a simple orbit x, the (total) dimension of the graded vector space $\mathrm{HC}_*(x^\kappa)$ is bounded by a constant independent of κ, provided that x^κ remains isolated.

(LC4) Assume that c is the only point of $\mathcal{S}(\alpha)$ in the interval $[a, b]$ and that all closed Reeb orbits with action c are isolated. (Hence, there are only finitely many such orbits.) Then

$$\mathrm{HC}_*^{(a,\,b)}(\alpha) = \bigoplus_{\mathcal{A}_\alpha(x)=c} \mathrm{HC}_*(x).$$

Here only (LC3) is not straightforward to prove. The assertion (LC1) follows from the facts that

$$|\mu_{CZ}(x) - \Delta(x)| \leq n - 1$$

for any closed Reeb orbit on a $(2n-1)$-dimensional contact manifold and that $\Delta(x)$ depends continuously on x; see, e.g., [52, 62]. The property (LC2) is an immediate consequence of the definition, and (LC4) follows from the fact that a holomorphic curve in the symplectization with zero ω-energy must be the cylinder over a Reeb orbit; see, e.g., [7, Lemma 5.4]. Finally, (LC3) is a far reaching generalization a theorem of Gromoll and Meyer, [30]. This result is established in [41, Section 6] as a consequence of a theorem from [26], asserting a similar upper bound for local Floer homology. (Note that (LC3) also follows from [28, Theorem 3 and Corollary 1] stated in Example 3.2.4 above. An analogue of (LC3) for (non-equivariant) symplectic homology is proved in [55].)

Proof of Theorem 3.2.2

In the proof, we will focus on the case of χ^+; for its negative counterpart, χ^-, can be handled in a similar fashion. Thus let $x_1, \ldots, x_r$ be the orbits of α with positive mean index: $\Delta(x_i) > 0$. Set $l_+ = 2n - 3$. By (LC1), this is the lowest degree for which the orbits with $\Delta = 0$ cannot contribute to the homology.

Our first goal is to prove (3.6) and (CH). To this end, observe that we have the following version of the Morse inequalities:

$$\dim \mathrm{HC}_l(\alpha) \leq \sum \dim \mathrm{HC}_l(x), \tag{3.13}$$

where, when $l \geq l_+$, the sum is taken over all orbits $x = x_i^\kappa$ with $\Delta(x) > 0$. This is a consequence of (LC4) and the long exact sequence for filtered contact homology. Since the mean index is homogeneous, i.e., $\Delta(x) = \kappa\Delta(x_i)$, we see that when κ is large the orbits x_i^κ do not contribute to $\mathrm{HC}_l(\alpha)$ or, to be more precise, to the right hand side of (3.13), due to (LC1) again. (It suffices to take $\kappa > (l+2)/\Delta(x_i)$ here.) In particular, the right hand side of (3.13) is finite when $l \geq l_+$. Furthermore, by (LC3), the contribution of x_i^κ to $\mathrm{HC}_l(\alpha)$ is bounded from above and the bound is independent of l. This proves (3.6): $\dim \mathrm{HC}_l(\alpha) \leq \mathit{const}$, where the constant is independent of $l \geq l_+$.

Let us now prove (3.9) and, particular, the fact that $\chi^+(\xi)$ is defined. Set

$$\chi(x) = \sum_l (-1)^l \dim \mathrm{HC}_l(x).$$

Thus, in the notation of (LC2), $\chi(x) = I_\kappa(F)$ where $x = y^\kappa$ and F is the Poincaré return map of y. Next note that for any $a \notin \mathcal{A}(\alpha)$, we have

$$\sum_l (-1)^l \dim \mathrm{HC}_l^a(\alpha) = \sum_{\mathcal{A}_\alpha(x)<a} \chi(x), \tag{3.14}$$

where the sum is now over all closed Reeb orbits with action less than a, but not as in (3.13) over only the iterations of the orbits x_i. This is again an immediate consequence of (LC2) and (LC4) and the long exact sequence for filtered contact homology.

When the summation is restricted to the degrees $l_+ \leq l \leq N$, a variant of (3.14) still holds up to a bounded error:

$$\Big|\sum_{l=l_+}^{N} (-1)^l \dim \mathrm{HC}_l^a(\alpha) - \sum \chi(x_i^\kappa)\Big| \leq const, \tag{3.15}$$

where *const* on the right is independent of a and N and the second sum is taken now over the orbits x_i^κ with $\mathcal{A}_\alpha(x_i^\kappa) < a$ and $\Delta(x_i^\kappa) \leq N$. Just as (3.13) and (3.14), this readily follows from (LC2) and (LC4) and the long exact sequence for filtered contact homology.

Let us divide (3.15) by N and let $a \to \infty$ and then $N \to \infty$. By Lemma 3.2.4, the first sum will then converge to $\chi^+(\xi)$, and hence

$$\chi^+(\xi) = \sum_i \lim_{N\to\infty} \frac{1}{N} \sum_{\Delta(x_i^\kappa)\leq N} \chi(x_i^\kappa),$$

provided that the limit on the right hand side exists. Since $\Delta(x_i^\kappa) = \kappa\Delta(x_i)$, we also have

$$\sum_{\Delta(x_i^\kappa)\leq N} \chi(x_i^\kappa) = \frac{1}{\Delta(x_i)} \sum_{\kappa=1}^{N} \chi(x_i^\kappa) + O(1),$$

as $N \to \infty$. As a result, by the definition of $\sigma(x_i)$ and (LC2),

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\Delta(x_i^\kappa)\leq N} \chi(x_i^\kappa) = \frac{\sigma(x_i)}{\Delta(x_i)},$$

which concludes the proof of the theorem. □

Remark 3.2.5. To guarantee the existence of the positive/negative MEC and prove (3.6) for positive/negative range, it suffices to assume only that the collection of simple Reeb orbits with positive/negative mean index is finite.

Remark 3.2.6. By (LC2) and Theorem 3.1.7, the function $\kappa \mapsto \chi(x^\kappa)$ is subordinated to the collection $\mathcal{N}(F)$ associated with the Poincaré return map F of x as in Section 3.1.2. Furthermore, recall that the local MEC of x is defined in [41] as

$$\hat{\chi}(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{\kappa=1}^{N} \chi(x^\kappa).$$

By (LC2), we have $\hat{\chi}(x) = \sigma(x)$. (In particular, the limit in the definition of $\hat{\chi}(x)$ exists.) With these observations in mind, our Theorem 3.2.2 can be easily obtained as a consequence of [41, Theorem 1.5].

3.2.5 Asymptotic Morse inequalities

The argument from the previous section lends itself readily to a proof of the asymptotic Morse inequalities. Namely, in the setting of Section 3.2.1, assume that ξ satisfies (HC) and set

$$\beta^{\pm}(\xi) = \limsup_{N\to\infty} \frac{1}{N} \sum_{l=l_\pm}^{N} \dim \mathrm{HC}_{\pm l}(\xi).$$

Likewise, when x is a simple closed Reeb orbit such that all iterations x^κ are isolated, set

$$\beta(x) = \limsup_{N\to\infty} \frac{1}{N} \sum_{\kappa=1}^{N} \dim \mathrm{HC}_*(x^\kappa).$$

This is finite number by (LC3); in fact, one can expect that in this case the limit exists; cf. Remark 3.2.8.

Theorem 3.2.7. *Assume that the Reeb flow of α on M^{2n-1} has only finitely many simple periodic orbits $x_1, \dots, x_r$, not necessarily non-degenerate, in the collection Γ. Then $\beta^{\pm}(\xi)$ is finite and moreover*

$$\beta^{\pm}(\xi) \leq \sum\nolimits^{\pm} \frac{\beta(x_i)}{\Delta(x_i)},$$

where $\sum^{\pm}$ stands for the sum taken over the orbits x_i with positive/negative mean index $\Delta(x_i)$.

This theorem can be proved exactly in the same way as Theorem 3.2.2. The non-degenerate case of this result is pointed out in [29, Remark 1.10]. Note that, as an immediate consequence of Theorem 3.2.7, we obtain a theorem from [41] (cf. [30, 55]) asserting that α must have infinitely many closed Reeb orbits when $\dim \mathrm{HC}_l(\xi)$ is unbounded as a function of $\pm l \geq l_{\pm}$.

Remark 3.2.8. At this stage, very little is known about the sequence $\dim \mathrm{HC}_*(x^{\kappa})$, except that this sequence is bounded, or about the "mean Betti number" $\beta(x)$. For instance, recall that the sequence $\chi(x^{\kappa}) = I_{\kappa}(F)$ is periodic by Theorem 3.1.7 and (LC2), and hence $\sigma(x)$ is rational. Moreover, a similar result holds for the local Floer homology. In fact, as is easy to see from the proof of [26, Theorem 1.1], the sequence $\dim \mathrm{HF}_*(x^{\kappa})$, where x is an isolated periodic orbit of a Hamiltonian diffeomorphism, is still subordinated to $\mathcal{N}$. We conjecture that this is also true for $\dim \mathrm{HC}_*(x^{\kappa})$, and hence $\beta(x) \in \mathbb{Q}$. However, neither of these facts have been proved yet.

Remark 3.2.9 (Variations). Variants of Theorems 3.2.2 and 3.2.7 hold for cylindrical contact homology, when the latter is defined, with straightforward modifications to the setting and virtually the same proofs; cf. Remark 3.2.1.

One can also expect the analogues of these theorems to hold for, say, the (positive) equivariant symplectic homology or for the S^1-equivariant Morse homology of the energy functional on the space of loops on Riemannian or Finsler manifold, or for the equivariant Morse-type homology associated with certain other functionals on loop spaces; see [9, 19, 51, 59, 72] for some relevant results. However, now the situation is more delicate because the corresponding complexes are not directly, by definition, generated by the (good) orbits and, strictly speaking, even in the non-degenerate case an extra step in the proof is needed to relate the homology and the orbits. For instance, in the non-degenerate analogue of Theorem 3.2.2 for either of these "homology theories", the contribution of an orbit x^κ is equal to the Euler characteristic of the infinite lens space $S^{2\infty-1}/\mathbb{Z}_\kappa$ with respect to a certain local coefficient system; cf. [4, Lecture 3]. This contribution is trivial when x^κ is bad and equal to ± 1 when the orbit is good. With this in mind, the proof of Theorem 3.2.2 should go through with mainly notational changes except for an analogue of (LC3), which now requires a proof; cf. [41] and [55].

3.3 Closed Reeb orbits on S^3

Our goal in this section is to reprove the following

Theorem 3.3.1 ([11, 28, 50]). *Let α be a contact form on S^3 such that $\ker\alpha$ is the standard contact structure. Then the Reeb flow of α has at least two closed orbits.*

In fact, the result proved in [11] is much more general and holds for all closed contact 3-manifolds; the proof uses the machinery of embedded contact homology. The proofs in [28, 50] both rely on a theorem establishing the "degenerate case of the Conley conjecture" for contact forms and more specifically asserting that the presence of one closed Reeb orbit of a particular type (the so-called SDM) implies the existence of infinitely many closed Reeb orbits. This result holds in all dimensions and is stated explicitly below. Another non-trivial, and in this case strictly three-dimensional, counterpart of the argument in [28] comes from the theory of finite energy foliations (see [37, 38]), while the argument in [50] uses, also in a non-trivial way, a variant of the resonance relation formula from [51]. Here, we give a very simple proof of Theorem 3.3.1 establishing it as a consequence of our Theorem 3.2.2 and the "Conley conjecture" type result mentioned above.

To state this result, we need to recall some definitions. Let F_t, $t \in [0, 1]$, be a family of germs of Hamiltonian diffeomorphisms of $\mathbb{R}^{2n}$ fixing the origin $0 \in \mathbb{R}^{2n}$ and generated by the germs of Hamiltonians H_t depending on $t \in S^1$. Set $F = F_1$ and let us assume that the origin is an isolated fixed point of F.

Denote by $\Delta(F) \in \mathbb{R}$ and $\mathrm{HF}_*(F)$ the mean index and, respectively, the local Floer homology of F over $\mathbb{Q}$. Again, we refer the reader to [23, 26, 55] for the definition of the latter. Here we only note that both of these invariants depend actually on the homotopy type (with fixed end points) of the entire family F_t. However, changing this family to a non-homotopic one, results in a shift of $\Delta(F)$ and of the grading of $\mathrm{HF}_*(F)$ by the same even number. Furthermore, $\mathrm{HF}_*(F)$ is supported in the range $[\Delta(F) - n, \Delta(F) + n]$, i.e., $\mathrm{HF}_*(F)$ vanishes for degrees outside this range. When $2n = 2$, $\mathrm{HF}_*(F)$ is supported in at most only one degree; [26]. Finally, analogously to (LC2) and essentially by definition, we have

$$I(F) = (-1)^n \sum (-1)^l \dim \mathrm{HF}_l(F).$$

The fixed point of F, the origin, is said to be a *symplectically degenerate maximum* (SDM) of F if $\Delta(x) \in \mathbb{Z}$ and $\mathrm{HF}_*(F)$ does not vanish at the upper boundary of its possible range, i.e., $\mathrm{HF}_{\Delta(x)+n}(F) \neq 0$. This is clearly a feature of the time-one map F, independent of the homotopy type of the family F_t. Note also that in this case F is necessarily totally degenerate, i.e., all eigenvalues of DF are equal to 1, although DF need not be the identity. (Clearly, $\Delta(F) \in \mathbb{Z}$ when F is totally degenerate; the converse, however, is not true without homological requirements.) Furthermore, then $\mathrm{HF}_{\Delta(x)+n}(F) = \mathbb{Q}$ and $\mathrm{HF}_*(F) = 0$ in all other degrees. (See [23, 26] for this and other results and a detailed discussion of SDM's.)

Example. Assume that H on $\mathbb{R}^{2n}$ is autonomous, has an isolated critical point at the origin and this critical point is a local maximum, and that the Hessian d^2H at the origin is identically zero. (More generally, it suffices to assume that the eigenvalues of d^2H with respect to the symplectic form are all zero.) Then the origin is an SDM of the time-one map F generated by H.

Returning to Reeb flows, we say that an isolated closed Reeb orbit x is an SDM if the corresponding fixed point of the Poincaré return map F of x is an SDM. (This is equivalent to requiring that $\Delta(x) \in \mathbb{Z}$ and $\mathrm{HC}_l(x)$ is $\mathbb{Q}$ when $l = \Delta(x) + 2n - 4$ and zero otherwise; see [28] for this and other related results.) Note that, although the grading of $\mathrm{HF}_*(x)$ depends on some extra choices, e.g., a trivialization of ξ along x, the notion of SDM is independent of these choices.

In the setting of Section 3.2.1, we have

Theorem 3.3.2 ([28]). *Let $(M^{2n-1}, \ker\alpha)$ be a contact manifold admitting a strong exact symplectic filling (W, ω) such that $c_1(TW) = 0$. Assume that the Reeb flow of α has an isolated simple closed Reeb orbit x which is an SDM. Then the Reeb flow of α has infinitely many periodic orbits.*

Remark 3.3.3. The proof of Theorem 3.3.2 is a straightforward, although lengthy and cumbersome, adaptation of the proof of the degenerate case of the Conley conjecture in [24]. In fact, in [28], the theorem is established under somewhat less restrictive conditions. Namely, it suffices to require that $\omega|_{\pi_2(W)} = 0$, when x is contractible (rather than that ω is exact) and that ω is atoroidal in general; cf. Remark 3.2.3.

Replacing the upper boundary of the range by the lower boundary in the definition of an SDM, we arrive at the notion of a *symplectically degenerate minimum* (SDMin). Thus F has an SDMin at the origin if and only if $\Delta(F) \in \mathbb{Z}$ and $\mathrm{HF}_{\Delta(F)-n}(F) \neq 0$. This notion naturally arises in dynamics (see [34] and also [28, Remark 1]), and Theorem 3.3.2 holds with an SDM replaced by an SDMin.

Just as the local Floer homology in dimension two, the local contact homology of an isolated closed Reeb orbit x on a 3-manifold is concentrated in at most one degree. Hence, in this case we have the following mutually exclusive possibilities when x is isolated and degenerate, but not necessarily simple:

- SDM: $\mathrm{HC}_{\Delta(x)}(x) \neq 0$ and x is an SDM,

- SDMin: $\mathrm{HC}_{\Delta(x)-2}(x) \neq 0$ and x is an SDMin,
- Saddle or "Monkey Saddle": $\mathrm{HC}_{\Delta(x)-1}(x) \neq 0$,
- Homologically Trivial: $\mathrm{HC}_*(x) = 0$.

All of these cases do occur. Furthermore, for any isolated iterated closed Reeb orbit y^κ (not necessarily non-degenerate), we have

$$(-1)^l \dim \mathrm{HC}_l(y^\kappa) = I_\kappa(F), \tag{3.16}$$

where F is the Poincaré return map of y and l is the degree where the cohomology of y^κ is supported. (Note that $\mathrm{HC}_*(y^\kappa) = 0$ if and only if $I_\kappa(F) = 0$.) This is an immediate consequence of (LC2) and again the fact that the cohomology is supported in only one degree.

Proof of Theorem 3.3.1. Arguing by contradiction, assume that the Reeb flow of α on (S^3, ξ_0) has only one closed simple orbit x. We will prove that then x is an SDM, and hence the flow has infinitely many periodic orbits by Theorem 3.3.2. Applying Theorem 3.2.2 and Example 3.2.2, we have

$$\frac{\sigma(x)}{\Delta(x)} = \frac{1}{2}, \tag{3.17}$$

where $\Delta(x) > 0$. Therefore, $\sigma(x) > 0$.

Furthermore, recall that $\mathrm{HC}_l(\xi_0) = \mathbb{Q}$ when $l \geq 2$ is even and $\mathrm{HC}_l(\xi_0) = 0$ otherwise; see, e.g., [6].

The orbit x is necessarily elliptic: all Floquet multipliers, i.e., eigenvalues of DF, lie on the unit circle. Indeed, if x is hyperbolic and even, we have $\sigma(x) = -1$, which is impossible. If x is hyperbolic and odd, we have $\sigma(x) = 1/2$ and $\Delta(x) = 1$. Furthermore, all orbits x^κ are non-degenerate and $\mu_{CZ}(x^\kappa) = \kappa$. Since the even iterations of x are bad orbits, the complex $\mathrm{CC}_*(\alpha)$ is generated by the orbits x^κ, where κ is odd, with $|x^\kappa| = \kappa - 1$, which is also impossible since then $\mathrm{HC}_0(\xi_0) = \mathbb{Q}$.

With x elliptic, write $\Delta(x) = 2m + 2\alpha$, where m is a non-negative integer and $\alpha \in [0, 1)$. By the definition of the mean index, $e^{\pm 2\pi i\alpha}$ is an eigenvalue of F. From (3.17) we see that α is necessarily rational. Set $\alpha = p/q$, where $q \geq 2$ and $q > p \geq 0$, and p and q are mutually prime. (In particular, x^q is degenerate.) To summarize, we are in the setting of Example 3.1.2: $I_\kappa(F) = 1 - r$ for some $r \geq 0$ when $q|\kappa$, and $I_\kappa(F) = 1$ otherwise. We conclude that

$$\sigma(x) = 1 - \frac{r}{q} \text{ and } \Delta(x) = 2m + \frac{2p}{q},$$

and (3.17) amounts to

$$1 - \frac{r}{q} = m + \frac{p}{q},$$

where m, r and p are non-negative and $q \geq 2$. This condition can only be satisfied when $m = 0$ or $m = 1$.

Let us first examine the case $m = 0$, which is exactly the point where in [28] we had to rely on the results from [37, 38]. Then $p > 0$ (for otherwise $\Delta(x) = 0$) and $p + r = q$. In particular, $r < q$.

To rule out this case, observe first that $p = 1$. Indeed, we have $|x^\kappa| = 0$ as long as $\kappa p/q < 1$. If $p \geq 2$, the first degree jump occurs when $\kappa p/q$ becomes greater than 1 and the degree increases to 2. The next jumps occur when $\kappa p/q$ moves over the subsequent integers, while x^κ is still nondegenerate, or when $\kappa = q$. In the latter case, x^q is degenerate, but $\Delta(x^q) \geq 4$ since $\kappa \geq 2$ and $p \geq 2$, and x^q cannot contribute to the homology in degree 0. We conclude that again $\mathrm{HC}_0(\alpha) \neq 0$ when $p > 1$, which is impossible.

Thus $p = 1$. The degrees of x^κ form a sequence

$$|x^\kappa| = \underbrace{0, \ldots, 0}_{q-1}, *, \underbrace{2, \ldots, 2}_{q-1}, *, \ldots,$$

where the undefined degrees of the degenerate iterations $|x^\kappa|$ are entered as the asterisks. It is clear, however, that $\mathrm{HC}_*(x^q)$ must be concentrated in degree one, to cancel the contribution of previous iterations to degree zero, and hence $r \geq 1$.

It follows from (3.16) that $\mathrm{HC}_*(x^\kappa) = \mathbb{Q}^{r-1}$, concentrated in degree $2\kappa/q - 1$, when $q|\kappa$ and that $\mathrm{HC}_*(x^\kappa) = \mathbb{Q}$, concentrated in degree $2\lfloor \kappa/q \rfloor$, otherwise, i.e., when x^κ is non-degenerate. Hence, all iterates x^κ with $\kappa \leq q-1$ contribute to degree 0 while the iterates x^κ with $\kappa > q$ contribute to the degrees greater than or equal to 2:

$$\mathrm{HC}_*(x^\kappa) = \underbrace{\mathbb{Q}_0, \ldots, \mathbb{Q}_0}_{q-1}, \mathbb{Q}_1^{r-1}, \underbrace{\mathbb{Q}_2, \ldots, \mathbb{Q}_2}_{q-1}, \mathbb{Q}_3^{r-1}, \ldots,$$

where the subscripts on the right hand side indicate the degree. To ensure that $\mathrm{HC}_0(\alpha) = 0$, we must have $q - 1 \leq r - 1$, by the long exact sequence for filtered contact homology, i.e., $q \leq r$, which is impossible since $r < q$ as stated above.

It follows now that $m = 1$ and $p = 0 = r$. Therefore, the orbit x itself is degenerate and $\Delta(x) = 2$. Clearly, since $\mathrm{HC}_*(x)$ is concentrated in at most one degree, the fact that $\mathrm{HC}_2(\xi_0) = \mathbb{Q}$ implies that $\mathrm{HC}_2(x) = \mathbb{Q}$, and hence x is an SDM. This concludes the proof of the theorem. $\square$

Remark 3.3.4. Let α be a contact form on (S^3, ξ_0) invariant under a contact involution τ. (For instance, this is the case when α comes from a symmetric star-shaped embedding of S^3 into $\mathbb{R}^4$ and τ is the antipodal involution or when α is the lift of a contact form on $\mathbb{RP}^3$ supporting the standard contact structure.) Then the Reeb flow of α cannot have exactly two simple Reeb orbits "swapped" by τ. Indeed, assume the contrary and denote the orbits by x and $\tau(x)$. Then these orbits have the same local invariants: the same mean index, the same mean iterated index and the same local contact homology. Now, arguing as in the proof of Theorem 3.3.1, but without relying on Theorem 3.3.2, it is not hard to show that this is impossible. However, as the example of an irrational ellipsoid in $\mathbb{R}^4$ shows, α can have exactly two simple orbits each of which is invariant under τ. Projecting the irrational ellipsoid contact form to $\mathbb{RP}^3 = ST^*S^2$, we obtain an asymmetric Finsler metric on S^2 with exactly two geodesics; see [43, 78] and also [32].

As an easy application of Theorem 3.3.1, we have

Corollary 3.3.5. *Let M be a fiberwise star-shaped (with respect to the zero section) hypersurface in T^*S^2. Then M carries at least two closed characteristics.*

When M is fiberwise convex, the Reeb flow is just the geodesic flow of a (not necessarily symmetric) Finsler metric on S^2. Thus, in particular, it follows that any Finsler metric on S^2 has at least two simple closed geodesics. This result is originally proved in [2]. Of course, Corollary 3.3.5 also immediately follows from the main theorem of [11].

Proof. The hypersurface M is of contact type and contactomorphic to $\mathbb{RP}^3 = ST^*S^2$

with the standard contact structure ξ_0 and some contact form α. Arguing by contradiction, assume that (M, α) carries only one simple closed Reeb orbit x. We claim that x is not contractible.

This is a standard fact which can be established in a variety of ways. For instance, this is a consequence of Remark 3.3.4. Alternatively, this follows from the fact that the Reeb flow of any such form α must have non-contractible orbits because the cylindrical contact homology of ST^*S^2 is non-zero for the non-trivial free homotopy type of loops in $\mathbb{RP}^3$. The latter assertion can be easily proved by examining the indices of the closed orbits for the form coming from an irrational ellipsoid in $\mathbb{R}^4$ or by using the Morse–Bott techniques; see, e.g., [6, 69, 45]. (For Finsler metrics on S^2 this is, of course, a well-known result in the standard calculus of variations: the orbit in question is the energy minimizer within the fixed non-trivial free homotopy class; cf. [4].)

Finally, since x is not contractible, it lifts to one orbit on S^3 and we obtain a contact form on S^3 with only one closed Reeb orbit, which is impossible by Theorem 3.3.1. $\square$

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