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**FILTERED SYMPLECTIC HOMOLOGY OF PREQUANTIZATION
BUNDLES AND THE CONTACT CONLEY CONJECTURE**

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Abstract

Filtered symplectic homology of prequantization bundles and the contact Conley conjecture

by

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In this thesis, we study Reeb dynamics on prequantization circle bundles and the filtered (equivariant) symplectic homology of prequantization line bundles, aka negative line bundles, with symplectically aspherical base. We define (equivariant) symplectic capacities, obtain an upper bound on their growth, prove uniform instability of the filtered symplectic homology and touch upon the question of stable displacement. We also introduce a new algebraic structure on the positive (equivariant) symplectic homology capturing the free homotopy class of a closed Reeb orbit – the linking number filtration – and use it to give a new proof of the non-degenerate case of the contact Conley conjecture (i.e., the existence of infinitely many simple closed Reeb orbits), not relying on contact homology.

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Chapter 1

Introduction

The main topic of this thesis is Reeb dynamics on prequantization S^1 -bundles and the filtered (equivariant) symplectic homology of the associated prequantization line bundles E , aka negative line bundles, over symplectically aspherical manifolds. The new features in this case, as compared to the symplectic homology of exact fillings, come from the difference between the Hamiltonian action, giving rise to the action filtration of the homology, and the contact action. We define equivariant symplectic capacities, obtain an upper bound on their growth, prove uniform instability of the filtered symplectic homology, and touch upon the question of stable displacement in E . We also introduce a new algebraic structure on the positive (equivariant) symplectic homology capturing the free homotopy class of a closed Reeb orbit – the linking number filtration. We then use this filtration to give a new proof of the non-degenerate case of the contact Conley conjecture, not relying on contact homology.

Prequantization S^1 -bundles M form an interesting class of examples to study the Reeb dynamics, and, in particular, the question of multiplicity of closed Reeb orbits with applications, for instance, to closed magnetic geodesics and geodesics on CROSS's; see, e.g., [27]. The range of possible dynamics behavior in

this case is similar to that for Hamiltonian diffeomorphisms and more limited than for all contact structures where it should more adequately be compared with the class of symplectomorphisms; see [24]. Furthermore, in many instances, various flavors of Floer-type homology groups associated to M can be calculated explicitly providing convenient basic tools to study the multiplicity questions.

Most of the Floer theoretic constructions counting closed Reeb orbits require a strong symplectic filling W of M and, in general, the properties of the resulting groups depend on the choice of W unless W is exact and $c_1(TW) = 0$. (The exceptions are the cylindrical contact homology and the contact homology linearized by an augmentation, but here we are only concerned with symplectic homology.) The most natural filling W of a prequantization S^1 -bundle M is that by the disk bundle or, to be more precise, by the region bounded by (M, α) , where α is the contact form, in E .

However, the filling W is also quite awkward to work with. The main reason is that W is never exact, although it is aspherical when the base B is aspherical. As a consequence, the Hamiltonian and contact actions of closed Reeb orbits differ and the Hamiltonian action, giving rise to the action filtration on the homology, is not necessarily non-negative; see Proposition 4.2.3. The second difficulty comes from that the natural map $\pi_1(M) \rightarrow \pi_1(W)$ fails, in general, to be one-to-one. This is the case, for instance, when B is symplectically aspherical: the fiber, which is not contractible in M , becomes contractible in W . This fact has important conceptual consequences. For instance, the proof of the contact Conley conjecture for prequantization bundles with aspherical base (i.e., the existence of infinitely many simple closed Reeb orbits) from [26, 27], which relies on the free homotopy class grading of the cylindrical contact homology, fails to directly translate to the symplectic homology framework.

One of the goals of this thesis is to systematically study the filtered symplectic homology of W , equivariant and ordinary. Global Floer theoretic “invariants” such as the equivariant and/or positive symplectic homology of E has been calculated explicitly; see [27, 44, 50]. Moreover, in many cases the global symplectic homology of E vanishes and this fact alone is sufficient for many applications.

The thesis is organized as follows. In Chapter 2 and 3, we set our conventions and notation and recall the constructions of various flavors of symplectic homology. The only non-standard point here is the definition of the negative symplectic homology. Namely, since the filling is not required to be exact, this homology cannot be defined as the subcomplex generated by the orbits with negative action. Instead, following [10], it is defined essentially as the homology of the subcomplex generated by the constant one-periodic orbits of an admissible Hamiltonian. In Chapters 4-6 we turn to new result. In Section 4.1, we investigate the consequences of vanishing of the global symplectic homology. We introduce a class of (equivariant) symplectic capacities, prove upper bounds on their growth, show that vanishing is equivalent to a seemingly stronger condition of uniform instability, and revisit the relation between vanishing of the symplectic homology and displacement. In Section 4.2, we specialize these results to prequantization line bundles with symplectically aspherical base and also briefly touch upon the question of stable displacement in prequantization bundles. A new algebraic structure on the positive (equivariant) symplectic homology of such bundles – the linking number filtration – is introduced in Chapter 5, where we also calculate the associated graded homology groups. This filtration is essentially given by the linking number of a closed Reeb orbit with the base B . It is then used in Chapter 6 to reprove the non-degenerate case of the contact Conley conjecture circumventing the foundational difficulties inherent in the construction of the contact homology.

Chapter 2

Contact geometry

2.1 Contact structures

In this section we recall the definition of contact structures and provide examples of contact manifolds.

2.1.1 Contact structures

Definition 2.1.1. Let M be a smooth manifold with dimension $2n + 1$. A *contact structure* on M is a smooth field ξ of tangent hyperplanes such that ξ is locally defined as $\xi = \ker \alpha$ for a 1-form α which satisfying $\alpha \wedge (d\alpha)^n \neq 0$. The pair (M, ξ) is called a *contact manifold* and α is called a *local contact form*.

If a contact form α is globally defined, $\alpha \wedge (d\alpha)^n$ is a volume form on M and thus M is orientable.

Remark 2.1.2. A locally defining 1-form α is not unique because $\ker \alpha = \ker(f\alpha)$ for a non-vanishing smooth function $f : M \rightarrow \mathbb{R}$.

Remark 2.1.3. The form $d\alpha|_{\xi}$ on ξ is nondegenerate (i.e., symplectic). Indeed, let $\{x_1, \dots, x_n, y_1, \dots, y_n, z\}$ be a local trivialization of TM such that $\ker \alpha =$

$\text{span}\{x_1, \dots, x_n, y_1, \dots, y_n\}$. Then

$$\alpha \wedge (d\alpha)^n(x_1, \dots, x_n, y_1, \dots, y_n, z) = \alpha(z)(d\alpha)^n(x_1, \dots, x_n, y_1, \dots, y_n).$$

Since $\alpha(z)$ is not zero, $d\alpha$ is not zero on ξ .

Definition 2.1.4. For two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) , a diffeomorphism $\phi : M_1 \rightarrow M_2$ is called a *contactomorphism* if

$$\phi_*(\xi_1) = \xi_2.$$

Remark 2.1.5. A contactomorphism does not usually preserve contact forms. It is possible to happen that $\phi^*(\alpha_2) = f\alpha_1$ for a non-vanishing function $f : M \rightarrow \mathbb{R}$.

Theorem 2.1.6. (Darboux) *Let (M, ξ) be a contact manifold. For a point $p \in M$ there exists a coordinate chart $(\mathcal{U}, x_1, \dots, x_n, y_1, \dots, y_n, z)$ centered at p such that on \mathcal{U} the 1-form*

$$\alpha = \sum_{i=1}^n x_i dy_i + dz$$

is a contact form for ξ .

Theorem 2.1.7. (Gray) *Let M be a closed contact manifold. Assume that α_t is a smooth family of global contact forms on M for $t \in [0, 1]$. Let ξ_t be the contact structure defined by α_t . Then there exists an isotopy $\rho : M \times [0, 1] \rightarrow M$ such that $\xi_t = (\rho_t)_*\xi_0$ for all $t \in [0, 1]$.*

2.1.2 Examples

Example 2.1.8. (Spheres)

$$S^{2n-1} = \left\{ (x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n} \mid \sum_{i=1}^n x_i^2 + y_i^2 = 1 \right\},$$

for $n \geq 1$. Consider the 1-form $\lambda = \frac{1}{2} \sum_{i=1}^n (x_i dy_i - y_i dx_i)$ on \mathbb{R}^{2n} and the inclusion $i : S^{2n-1} \hookrightarrow \mathbb{R}^{2n}$. The pull-back, $\alpha := i^* \lambda$ is a contact form on S^{2n-1} . To show that α is a contact form on S^{2n-1} , first consider the 1-form $\nu = d \left(\sum_{i=1}^n x_i^2 + y_i^2 \right)$ on \mathbb{R}^{2n} . Then $\ker \nu = TS^{2n-1}$. Let $\{v_1, \dots, v_{2n-1}\}$ be a basis of $T_p S^{2n-1}$. Extend the basis to a basis \mathbb{R}^{2n} , $\{u, v_1, \dots, v_{2n-1}\}$. Then

$$\nu \wedge \alpha \wedge (d\alpha)^{n-1}(u, v_1, \dots, v_{2n-1}) = \nu(u) (\alpha \wedge (d\alpha)^{n-1}(v_1, \dots, v_{2n-1})).$$

Hence, if $\nu \wedge \alpha \wedge (d\alpha)^{n-1}$ is not zero at every point of S^{2n-1} then $\alpha \wedge (d\alpha)^{n-1}$ is not zero on S^{2n-1} . By computation, one can see that

$$\nu \wedge \alpha \wedge (d\alpha)^{n-1} = \left(\sum_{i=1}^n x_i^2 + y_i^2 \right) (x_1 \wedge y_1 \wedge \dots \wedge x_n \wedge y_n).$$

Thus, S^{2n-1} is a contact manifold. The distribution $\ker \alpha$ is called the *standard contact structure* on S^{2n-1} .

Example 2.1.9. (Cotangent Sphere Bundle) Let M be a manifold with dimension n and T^*M the cotangent bundle of M . The cotangent sphere bundle is defined by

$$S(T^*M) := (T^*M \setminus 0) / \sim,$$

where $(p, \lambda) \sim (p, \lambda')$ if $\lambda = t\lambda'$ for some $t > 0$. Set $[\lambda] = \{\lambda' \in T^*M \mid \lambda = t\lambda' \text{ for some } t > 0\}$. Fix a local section $\sigma : S(T^*M) \rightarrow T^*M$ such that $\sigma(p, [\lambda]) = (p, \lambda)$. Then $(d\pi_{(p, [\lambda])})^*(\sigma(p, [\lambda]))$ is a contact form at $(p, [\lambda])$ in $S(T^*M)$, where $\pi : S(T^*M) \rightarrow M$ be the natural projection. Next, let $(x_0, \dots, x_{n-1}, \lambda_0, \dots, \lambda_{n-1})$ be local coordinates on T^*M . Then $\sigma(p, [\lambda])$ can be regarded as $\lambda_0 dx_0 + \dots + \lambda_{n-1} dx_{n-1}$ over a point $p \in M$. Consider a neighborhood of $(p, [\lambda])$ on which $\lambda_0 \neq 0$. Then on the neighborhood, $\ker d\pi^* \sigma = \ker (dx_0 + \sum_{i=1}^{n-1} \lambda_i dx_i)$.

Example 2.1.10. (Triple torus) Consider the torus \mathbb{T}^3 as $\mathbb{R}^3/\mathbb{Z}^3$. Then the 1-form $\alpha = \sin(2\pi kz)dx + \cos(2\pi kz)dy$ is a contact form on \mathbb{T}^3 for a nonzero integer k .

One can see that $S(T^*\mathbb{T}^2)$ is identified with \mathbb{T}^3 , where \mathbb{T}^2 is the double torus. When $k = 1$ this is the same as $S(T^*\mathbb{T}^2)$.

2.2 Contact dynamics

In this section we recall the definition of Reeb vector fields and briefly explore Reeb dynamics on prequantization line bundles.

2.2.1 Reeb vector fields

Definition 2.2.1. Let (M, ξ) be a contact manifold with a global contact form α . There exists a unique vector field R satisfying

$$\iota_R d\alpha = 0. \quad \text{and} \quad \alpha(R) = 1.$$

The vector field R is called the *Reeb vector field* determined by α .

Every vector field X on M can be uniquely written as $X = fR + Y$ for $Y \in \xi$ and $f : M \rightarrow \mathbb{R}$.

Remark 2.2.2. The flow of the Reeb vector field preserves the contact form α . Let ϕ_t be the flow.

$$\frac{d}{dt} \phi_t^*(\alpha) = \phi_t^*(\mathcal{L}_R \alpha) = \phi_t^*(d\iota_R \alpha + \iota_R d\alpha) = 0.$$

Hence, $\phi_t^*(\alpha) = \phi_0^*(\alpha) = \alpha$ for all $t \in \mathbb{R}$.

Definition 2.2.3. A *Reeb orbit* is a closed orbit of the Reeb vector field R determined by a contact form α . In other words, a Reeb orbit is a map $\gamma : S^1 \rightarrow M$ such that $\gamma'(t) = R(\gamma(t))$.

Example 2.2.4. (Spheres) Consider S^{2n-1} and the contact form α as it is in Example 2.1.8. The vector field $R = 2 \sum_{i=1}^n (x_i \partial_{y_i} - y_i \partial_{x_i})$ is the Reeb vector field determined by α . There are n simple Reeb orbits which are circles. For the sphere all orbits are closed.

Example 2.2.5. (Triple torus) Consider \mathbb{T}^3 and the contact form α as it is in Example 2.1.10. The Reeb vector field is $R = \sin(2\pi kz) \partial_x + \cos(2\pi kz) \partial_y$.

Consider an action functional \mathcal{A}_α on $C^\infty(S^1, M)$ defined by

$$\mathcal{A}_\alpha(\gamma) = \int_\gamma \alpha.$$

Then γ is a critical point of \mathcal{A}_α if and only if γ is a Reeb orbit of R with the period $\mathcal{A}_\alpha(\gamma)$ [4]. The action $\mathcal{A}_\alpha(\gamma)$ is called the *contact action* of γ . Denote by $\mathcal{S}(\alpha)$ the set $\{\mathcal{A}_\alpha(\gamma) \mid \gamma \text{ is a Reeb orbit of } R\}$.

2.2.2 Symplectization

Proposition 2.2.6. *Let (M, ξ) be a contact manifold with a contact form α , and let $\widetilde{M} = M \times \mathbb{R}$. Then the 2-form $d(e^t \pi^* \alpha)$ is a symplectic form on \widetilde{M} , where $\pi : \widetilde{M} \rightarrow M$ is the natural projection and t is a coordinate of \mathbb{R} .*

The symplectic manifold $(\widetilde{M}, d(e^t \pi^* \alpha))$ is called the *symplectization* of M . This symplectic manifold is independent of the choice of contact forms.

Example 2.2.7. Consider S^{2n-1} as it is in Example 2.1.8. The symplectization of S^{2n-1} is symplectomorphic to $(\mathbb{R}^{2n} \setminus \{0\}, \sum dx_i \wedge dy_i)$. Define a smooth map $\psi : \mathbb{R}^{2n} \setminus \{0\} \rightarrow S^{2n-1} \times \mathbb{R}$ by

$$\psi(p) = \left(\frac{p}{|p|}, \ln |p|^2 \right).$$

The map ψ is a diffeomorphism which has the inverse $\phi : S^{2n-1} \times \mathbb{R} \rightarrow \mathbb{R}^{2n-1} \setminus \{0\}$ defined by $\phi(p, t) = e^{t/2} \cdot p$. Next, show ψ preserves symplectic structures. By identifying $S^{2n-1} \times \mathbb{R}$ with $\mathbb{R}^{2n-1} \setminus \{0\}$, the natural projection $\pi : S^{2n-1} \times \mathbb{R} \rightarrow S^{2n-1}$ can be regarded as

$$\pi : \mathbb{R}^{2n-1} \setminus \{0\} \rightarrow S^{2n-1} \text{ defined by } \pi(p) = e^{-t/2} \cdot p.$$

Let $(x_1, y_1, \dots, x_n, y_n)$ be the standard coordinates on $\mathbb{R}^{2n-1} \setminus \{0\}$. Then $\pi^* \alpha = \frac{1}{2e^t} \sum_{i=1}^n (x_i dy_i - y_i dx_i)$. Hence,

$$d(e^t \pi^* \alpha) = d\left(\frac{1}{2} \sum_{i=1}^n (x_i dy_i - y_i dx_i)\right) = \sum_{i=1}^n dx_i \wedge dy_i.$$

2.2.3 Hypersurfaces of contact type

Definition 2.2.8. Let (W, ω) be a symplectic manifold. A *Liouville vector field* X is a vector field on W satisfying $\mathcal{L}_X \omega = \omega$.

Definition 2.2.9. A hypersurface M of a symplectic manifold (W, ω) is called a *hypersurface of contact type* if there exists a Liouville vector field X defined in a neighborhood of M and transverse to M .

Remark 2.2.10. If M is a hypersurface of contact type in a symplectic manifold (W, ω) , the 1-form $\iota_X \omega$ is a contact form on M . The 1-form is called the *Liouville form*.

Definition 2.2.11. Let (W, ω) be a compact symplectic manifold and M a contact type boundary with a Liouville vector field X . If X points outward along M , the manifold W is said to have a *convex boundary*. If X points inward, W is said to have a *concave boundary*.

Example 2.2.12. Consider the symplectic manifold $\left(\mathbb{R}^{2n}, \omega = \sum_{i=1}^n dx_i \wedge dy_i\right)$. The vector field $X = \frac{1}{2} \sum_{i=1}^n (x_i \partial_{x_i} + y_i \partial_{y_i})$ is a Liouville vector field pointing outward. One can see that

$$\iota_X \omega = \frac{1}{2} \sum_{i=1}^n (x_i dy_i - y_i dx_i)$$

$$\mathcal{L}_X \omega = d\iota_X \omega = \sum_{i=1}^n dx_i \wedge dy_i = \omega.$$

By restricting $\iota_X \omega$ to the sphere S^{2n-1} , we obtain the contact form on S^{2n-1} which we've seen in Example 2.1.8. Hence, S^{2n-1} is of contact type.

2.2.4 Strong symplectic filling

Definition 2.2.13. A *Liouville domain* is an exact symplectic manifold (W, ω) with a convex boundary. In other words, $\omega = d\lambda$ for some 1-form λ and the Liouville vector field X satisfying $\iota_X \omega = \lambda$ points outward along the boundary.

Example 2.2.14. (Balls) The n -ball defined by

$$B^n = \left\{ (x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n} \mid \sum_{i=1}^n x_i^2 + y_i^2 \leq 1 \right\}$$

is a Liouville domain with the boundary S^{2n-1} .

Definition 2.2.15. A *strong symplectic filling* of a contact manifold (M, ξ) is a symplectic manifold (W, ω) satisfying the following conditions:

- (i) M is the convex boundary of W ;
- (ii) $\xi = \ker(\iota_X \omega)$,

where X is the Liouville vector field.

As the Liouville vector field points outwards along M , the orientation of M agrees with the orientation $\alpha \wedge (d\alpha)^{n-1}$ when $\dim W = 2n$.

Let (W, ω) be a compact symplectic manifold with a contact type boundary (M, α) . Let X be a Liouville vector field defined on a neighborhood U of M , and ϕ the flow of X . Here, choose X such that the map $g : M \times [-\epsilon, 0] \rightarrow U$ defined by $(p, t) \mapsto \phi^t(p)$ satisfies $g^*(\iota_X \omega) = e^t \alpha$ and $g_* \partial_t = X$. Then $g^* \omega = d(e^t \alpha)$ and $d(e^t \alpha)$ is a symplectic form on $M \times [0, \infty)$ [47, 5].

Definition 2.2.16. The *symplectic completion* \widehat{W} of W is the union

$$\widehat{W} := W \cup_g M \times [0, \infty)$$

endowed with the symplectic form

$$\widehat{\omega} := \begin{cases} \omega & \text{on } W; \\ d(e^t \alpha) & \text{on } M \times [0, \infty). \end{cases}$$

For the rest of the thesis, we set $e^t = r$ and thus r is regarded as a coordinate on $[1, \infty)$. We use the expression $W \cup_M M \times [1, \infty)$ for $W \cup_g M \times [0, \infty)$.

2.3 Prequantization line bundles

Throughout this section let (B, σ) be a symplectic manifold.

Definition 2.3.1. A symplectic form σ is called *aspherical* if for any smooth map $f : S^2 \rightarrow B$,

$$\int_{S^2} f^* \sigma = 0.$$

The above condition can be written as $\sigma|_{\pi_2(B)} = 0$.

Definition 2.3.2. A closed form λ is called *integral* if for any finite singular cocycle C with integer coefficient,

$$\int_C \lambda \in \pi\mathbb{Z}.$$

If a symplectic form σ is integral, we say the symplectic manifold (B, σ) is *integral*.

The set of all elements of $H^2(B, \mathbb{R})$ which contain integral closed forms is the image of the homomorphism $\psi : H^2(B, \pi\mathbb{Z}) \rightarrow H^2(B, \mathbb{R})$ induced by the natural homomorphism $\pi\mathbb{Z} \rightarrow \mathbb{R}$. Thus, σ is integral if and only if the cohomology class $[\sigma]$ is in the image of ψ . We refer to [38] for more details.

Definition 2.3.3. Let (B, σ) be an integral symplectic manifold. A *prequantization S^1 -bundle* M is a principal S^1 -bundle $\pi : M \rightarrow B$ with a connection form α_0 such that the curvature form of α_0 is $d\alpha = \pi^*\sigma$ when S^1 is regarded as \mathbb{R}/\mathbb{Z} .

Remark 2.3.4. In consequence of (iii) and the definition of the connection form, α_0 is a contact form of M , and the Reeb vector field of α_0 is a generator of the S^1 action on M . Also, the first Chern class of the circle bundle $c_1(M)$ is $-[\sigma]/\pi$ with suitable conventions; see [21].

Assume that (B^{2m}, σ) be a *symplectically aspherical* manifold, i.e., $\sigma|_{\pi_2(W)} = 0 = c_1(TB)|_{\pi_2(W)}$, such that σ is integral. Also, assume that $[\sigma] \in H^2(B, \pi\mathbb{Z})$, where π is the number $3.14\dots$. Let $\pi : M \rightarrow B$ be a prequantization S^1 -bundle and $W := M \times_{S^1} D^2 \rightarrow B$ be the disk bundle. The associated line bundle $\pi : E \rightarrow B$ is a symplectic manifold with symplectic form

$$\omega = \frac{1}{2}(\pi^*\sigma + d(r^2\alpha_0)),$$

where $r : E \rightarrow [0, \infty)$ is the fiberwise distance to the zero section. We call (E, ω)

a *prequantization line bundle*. Note that away from the zero section we have

$$\omega = \frac{1}{2}d((1+r^2)\alpha_0).$$

Let $\alpha = f\alpha_0$ be a contact form on M supporting $\ker \alpha_0$. Without loss of generality, we may assume that $f > 1/2$. Then the fiberwise star-shaped hypersurface given by the condition $(1+r^2)/2 = f$ is of contact type. Denote by M_f the hypersurface. The restriction of the primitive $(1+r^2)\alpha_0/2$ to M_f is exactly α . Denote by W_f the domain bounded by M_f in E . The domain W_f is a strong symplectic filling of M_f diffeomorphic to the associated disk bundle. We can identify E with the symplectic completion \widehat{W} of W .

Denote by $\tilde{\pi}_1(M)$ the collection of free homotopy classes of loops in M or equivalently the set of conjugacy classes in $\pi_1(M)$. Furthermore, let \mathfrak{f} be the free homotopy class of the fiber in M or in $E \setminus B$ and $\mathfrak{f}^{\mathbb{Z}} = \{\mathfrak{f}^k \mid k \in \mathbb{Z}\}$.

When σ is aspherical, the homotopy long exact sequence of the circle bundle $M \rightarrow B$ splits and we have

$$1 \rightarrow \pi_1(S^1) \rightarrow \pi_1(M) \rightarrow \pi_1(B) \rightarrow 1.$$

It is not hard to see that $\mathfrak{f}^{\mathbb{Z}}$ is the image of $\mathbb{Z} \cong \pi_1(S^1)$ in $\tilde{\pi}_1(M) = \tilde{\pi}_1(E \setminus B)$; [26, Lemma 4.1]. Furthermore, this is exactly the set of free homotopy classes of loops x with contractible projections to B , i.e., of loops contractible in E . The one-to-one correspondence $\mathfrak{f}^{\mathbb{Z}} \rightarrow \mathbb{Z}$ is given by the *linking number* $L_B(x)$ of x with B . This is simply the intersection index of a generic disk bounded by x with B .

Chapter 3

Symplectic homology

3.1 Symplectic homology

In this section we briefly recall the definition of symplectic homology and define negative and positive symplectic homology. We refer to other sources for a more detailed discussion; see, e.g., [5, 6, 9, 10, 14, 25, 56, 61]. Throughout the thesis all homology groups are taken with rational coefficients unless specifically stated otherwise. This choice of the coefficient field is essential, although suppressed in the notation, and some of the results are simply not true when, say, the coefficient field has finite characteristic.

3.1.1 Mean index and the Conley-Zehnder index of a path in $\mathrm{Sp}(2n; \mathbb{R})$

Throughout this subsection, we focus on paths $\Phi : [0, 1] \rightarrow \mathrm{Sp}(2n; \mathbb{R})$ such that $\Phi(0) = I$, where I is the identity matrix.

To define the mean index and the Conley-Zehnder index of a path in $\mathrm{Sp}(2n; \mathbb{R})$, we recall the following theorem and propositions.

Theorem 3.1.1. *There exists a continuous map $\rho : \mathrm{Sp}(2n; \mathbb{R}) \rightarrow S^1$ satisfying the following conditions:*

(i) *Naturality: For A and T in $\mathrm{Sp}(2n; \mathbb{R})$,*

$$\rho(TAT^{-1}) = \rho(A).$$

(ii) *Product: If $A \in \mathrm{Sp}(2m; \mathbb{R})$ and $B \in \mathrm{Sp}(2n; \mathbb{R})$,*

$$\rho \left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right) = \rho(A)\rho(B).$$

(iii) *Determinant: If $A \in \mathrm{Sp}(2n; \mathbb{R}) \cap O(2n) = U(n)$,*

$$\rho(A) = \det_{\mathbb{C}}(X + iY), \quad \text{where } A = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}.$$

(iv) *Normalization: If A has no eigenvalues on the unit circle,*

$$\rho(A) = \pm 1.$$

It follows from the property (iii) that ρ induces an isomorphism between $\pi_1(\mathrm{Sp}(2n; \mathbb{R}))$ and \mathbb{Z} .

The explicit construction of the continuous map ρ is described in [53].

Proposition 3.1.2. *The set of all symplectic matrices with distinct eigenvalues is dense in $\mathrm{Sp}(2n; \mathbb{R})$.*

Proposition 3.1.3. *The symplectic group $\mathrm{Sp}(2n; \mathbb{R})$ is connected.*

From Proposition 3.1.2 and 3.1.3, we see that every matrix $A \in \text{Sp}(2n; \mathbb{R})$ can be connected to a matrix $B \in \text{Sp}(2n; \mathbb{R})$ with distinct eigenvalues.

Proposition 3.1.4. *A matrix $A \in \text{Sp}(2n; \mathbb{R})$ with distinct eigenvalues can be written as the direct sum of matrices $A_j \in \text{Sp}(2; \mathbb{R})$ and matrices with complex eigenvalues not on the unit circle.*

Proposition 3.1.5. *Let $A \in \text{Sp}(2n; \mathbb{R})$. There are 3 types of eigenvalues of A :*

- (i) *If an eigenvalue λ is not on the unit circle nor a real number, then the quadruple $\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1}$ occurs.*
- (ii) *If an eigenvalue λ is on the unit circle, then the pair $\lambda, \bar{\lambda}$ occurs. Especially, if $\lambda = \pm 1$ then λ has even multiplicity.*
- (iii) *If an eigenvalue λ is a real number, then the pair λ, λ^{-1} occurs.*

Define a symplectic form ω_0 on \mathbb{C}^{2n} by

$$\omega_0(X, Y) = (J_0 X)^T Y,$$

where $J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$.

Definition 3.1.6. Let A be a matrix in $\text{Sp}(2n; \mathbb{R})$ and λ be an eigenvalue of A . The eigenvalue λ is called *an eigenvalue of first kind* if either of the followings is satisfied:

- (i) λ is on the unit circle, i.e., $\lambda = \exp(i\theta)$ and $\text{Im } \omega_0(\xi, \bar{\xi}) > 0$, where ξ is the eigenvector corresponding to λ ;
- (ii) λ is not on the unit circle and $|\lambda| < 1$.

By following the construction of ρ in [53], one can see the map ρ defined by

$$\rho(A) = \prod_{\lambda \text{ of first kind}} \frac{\lambda}{|\lambda|}.$$

By the definition of ρ , if A is a matrix in $\text{Sp}(2; \mathbb{R})$,

$$\rho(A) = \begin{cases} \exp(i\theta) & \text{if } A \text{ has an eigenvalue of first kind } \lambda = \exp(i\theta); \\ 1 & \text{if } A \text{ has positive eigenvalues;} \\ -1 & \text{if } A \text{ has negative eigenvalues.} \end{cases}$$

If A is a matrix in $\text{Sp}(4; \mathbb{R})$ and has complex eigenvalues not on the unit circle,

$$\rho(A) = 1.$$

If A is a matrix in $\text{Sp}(2n; \mathbb{R})$ with distinct eigenvalues, then by Proposition 3.1.4 and Theorem 3.1.1 (ii) and (iv), $\rho(A) = \prod_j \rho(A_j)$, where $A_j \in \text{Sp}(2; \mathbb{R})$. Considering all of the above, for a path Φ in $\text{Sp}(2n; \mathbb{R})$, we can find a continuous function $\theta(t)$ such that

$$\rho(\Phi(t)) = \exp(i\theta(t)).$$

Definition 3.1.7. The *mean index* of a path $\Phi : [0, 1] \rightarrow \text{Sp}(2n; \mathbb{R})$ is defined by

$$\hat{\mu}(\Phi) = \frac{\theta(1) - \theta(0)}{\pi}.$$

Definition 3.1.8. A path Φ in $\text{Sp}(2n; \mathbb{R})$ is said to be *nondegenerate* if the matrix $\Phi(1)$ does not have the eigenvalue 1, i.e., $\det(I - \Phi(1)) \neq 0$.

Let us define the Conley-Zehnder index of a nondegenerate path $\Phi \in \text{Sp}(2n; \mathbb{R})$.

Set

$$\mathrm{Sp}^*(2n; \mathbb{R}) = \{A \in \mathrm{Sp}(2n; \mathbb{R}) \mid \det(I - A) \neq 0\}.$$

Then $\mathrm{Sp}^*(2n; \mathbb{R})$ has two connected components:

$$\mathrm{Sp}^+(2n; \mathbb{R}) = \{A \in \mathrm{Sp}(2n; \mathbb{R}) \mid \det(I - A) > 0\}$$

and

$$\mathrm{Sp}^-(2n; \mathbb{R}) = \{A \in \mathrm{Sp}(2n; \mathbb{R}) \mid \det(I - A) < 0\}.$$

Definition 3.1.9. The *Conley-Zehnder index* of a nondegenerate path $\Phi \in \mathrm{Sp}(2n; \mathbb{R})$ is defined as follows:

Case 1. If $\Phi(1) \in \mathrm{Sp}^+(2n; \mathbb{R})$ then $\mu_{CZ}(\Phi) = \hat{\mu}(\Psi^+ \cdot \Phi)$, where Ψ^+ is a path connecting $\Phi(1)$ to $-I$ in its connected component. and $\Psi^+ \cdot \Phi$ is the concatenation of two paths.

Case 2. If $\Phi(1) \in \mathrm{Sp}^-(2n; \mathbb{R})$ then $\mu_{CZ}(\Phi) = \hat{\mu}(\Psi^- \cdot \Phi)$, where Ψ^- is a path connecting $\Phi(1)$ to $\begin{pmatrix} 2 & 0 & \mid & 0 \\ 0 & \frac{1}{2} & \mid & \\ \hline 0 & & \mid & -I \end{pmatrix}$ in its connected component. and $\Psi^- \cdot \Phi$ is the concatenation of two paths.

The Conley-Zehnder index has the following properties.

Proposition 3.1.10. *Let μ be the Conley-Zehnder index*

- (i) *The mean index and the Conley-Zehnder index are invariant under homotopies.*
- (i) *Let Φ be a nondegenerate path in $\mathrm{Sp}(2n; \mathbb{R})$. For any path Ψ in $\mathrm{Sp}(2n; \mathbb{R})$,*

$$\mu(\Psi\Phi\Psi^{-1}) = \mu(\Phi).$$

We briefly introduce the definition of the Robbin-Salamon index by following [52].

Definition 3.1.11. Let Φ be a smooth path in $\mathrm{Sp}(2n; \mathbb{R})$. Write $\dot{\Phi}(t) = J_0 S(t) \Phi(t)$, where $S : [0, 1] \rightarrow \{A \in M_{2n}(\mathbb{R}) \mid A^T = A\}$ is a path and J_0 is an almost complex structure compatible with the standard symplectic form $\omega_0 = \sum dx_i \wedge y_i$. A number $t \in [0, 1]$ is called a *crossing* if $\det(\Phi(t) - I) = 0$. The *crossing form* $\Gamma(\Phi, t)$ is defined as the quadratic form of $S(t)$ to $\ker(\Phi(t) - I)$. A crossing t_0 is called *regular* if the crossing form $\Gamma(\Phi, t_0)$ is nondegenerate.

Definition 3.1.12. The Robbin-Salamon index of a path Φ in $\mathrm{Sp}(2n; \mathbb{R})$ is defined by

$$\mu_{RS}(\Phi) = \frac{1}{2} \mathrm{sign} \Gamma(\Phi, 0) + \sum_{\substack{t \text{ crossing} \\ t \in [0, 1]}} \mathrm{sign} \Gamma(\Phi, t) + \frac{1}{2} \mathrm{sign} \Gamma(\Phi, 1),$$

where $\mathrm{sign} \Gamma(\Phi, t)$ is the signature of the crossing form.

Remark 3.1.13. In contrast to the Conley-Zehnder index, the Robbin-Salamon index is defined for degenerate paths in $\mathrm{Sp}(2n; \mathbb{R})$. Also, the Robbin-Salamon index is invariant under homotopies with fixed end points.

3.1.2 Hamiltonian action functional and the Conley-Zehnder index

Before defining symplectic homology, Let us briefly recall the definitions of the Hamiltonian action functional and the Conley-Zehnder index of a nondegenerate orbit. For the rest of this thesis, we regard S^1 as \mathbb{R}/\mathbb{Z} .

Let (W, ω) be an aspherical symplectic manifold and $H : S^1 \times W \rightarrow \mathbb{R}$ be a Hamiltonian. The vector field X_H satisfying $\iota_{X_H} \omega = -dH$ is called the *Hamiltonian vector field*. The flow of X_H is called the *Hamiltonian flow*. Denote by $\mathcal{P}(H)$ the collection of its closed orbits which are contractible in W .

Let Λ be the subspace of all contractible loops in $C^\infty(S^1, W)$. Define an action functional on Λ by

$$\mathcal{A}_H(x) = \int_{\bar{x}} \omega - \int_{S^1} H(t, x(t)) dt,$$

where $\bar{x} : D^2 \rightarrow W$ such that $x = \bar{x}|_{\partial D^2}$. We call this action functional the *Hamiltonian action functional* and the map \bar{x} a *capping* of x . Denote by $\mathcal{A}_\omega(x)$ the symplectic area bounded by the orbits, i.e., $\int_{\bar{x}} \omega$, and call it the *symplectic area* of x . By the assumption that ω is aspherical, this action functional is well-defined. The differential of \mathcal{A}_H at x is

$$(d\mathcal{A}_H)_x(Y) = \int_{S^1} \omega(Y(t), \dot{x}(t) - X_H(x(t))) dt.$$

The critical points of \mathcal{A}_H are elements in $\mathcal{P}(H)$.

Definition 3.1.14. Let ϕ_H^t be the Hamiltonian flow and x be a closed orbit of ϕ_H^t . If $\det(I - d_{x(0)}\phi_H^1) \neq 0$ then x is said to be *nondegenerate*.

Let x be a nondegenerate closed orbit which is contractible in W , and \bar{x} be a capping of x . Then we choose a symplectic trivialization for $\bar{x}^*(TW)$. Restrict the trivialization to $S^1 \times \mathbb{R}^{2n}$, and denote the restriction by Φ , i.e., $\Phi : S^1 \times \mathbb{R}^{2n} \rightarrow x^*(TW)$. Define a path $\Psi : [0, 1] \rightarrow \text{Sp}(2n, \mathbb{R})$ by

$$\Psi(t) = \Phi^{-1}(x(t)) \circ d_{x(0)}\phi_H^t \circ \Phi(x(0)).$$

Then $\Psi(0) = I$ and $\det(I - \Psi(1)) \neq 0$ by the nondegeneracy of x . Hence, the Conley-Zehnder index of the path Ψ is defined.

Definition 3.1.15. Assume that $c_1(TW)|_{\pi_2(W)} = 0$. Let x be a nondegenerate closed orbit which is contractible in W . The *Conley-Zehnder index* of x is defined

by

$$\mu(x) = \mu_{CZ}(\Psi).$$

By the property (ii) in Proposition 3.1.10, the Conley-Zehnder index is independent of the choice of a trivialization only for a fixed capping of x . Provided that $c_1(TW)|_{\pi_2(W)} = 0$, the Conley-Zehnder index is independent of the choice of the capping; see [53, Section 5].

Remark 3.1.16. When H is a C^2 -small autonomous Hamiltonian on W^{2n} , for a nondegenerate critical point x ,

$$\mu(x) = n - \mu_M(x),$$

where $\mu_M(x)$ is the Morse index of x .

3.1.3 Symplectic homology

Let (W, ω) be a strong symplectic filling of a contact manifold (M, ξ) such that $\xi = \ker \alpha$ for some 1-form α . Assume that $c_1(TW)|_{\pi_2(W)} = 0$. Consider the symplectic completion $\widehat{W} := W \cup_M M \times [1, \infty)$ with the symplectic form

$$\widehat{\omega} := \begin{cases} \omega & \text{on } W; \\ d(r\alpha) & \text{on } M \times [1, \infty), \end{cases}$$

where r is a coordinate for $[1, \infty)$.

We denote by $\mathcal{P}(\alpha)$ the set of Reeb orbits of α contractible in W , and denote by $\mathcal{S}_{\widehat{\omega}}(W)$ the set of the symplectic areas of orbits in $\mathcal{P}(\alpha)$, i.e.,

$$\mathcal{S}_{\widehat{\omega}}(W) = \left\{ \int_{\bar{x}} \widehat{\omega} \mid x \in \mathcal{P}(\alpha) \right\}.$$

When $\widehat{\omega}$ is exact, the symplectic area $\mathcal{A}_{\widehat{\omega}}(x)$ is the same as the contact action $\mathcal{A}_\alpha(x)$.

Consider a Hamiltonian $H : S^1 \times \widehat{W} \rightarrow \mathbb{R}$. Denote by $\mathcal{P}(H)$ the set of one-periodic orbits of H contractible in \widehat{W} . Define the *action spectrum* as

$$\mathcal{S}(H) := \{\mathcal{A}_H(x) \mid x \in \mathcal{P}(H)\}.$$

Definition 3.1.17. A Hamiltonian $H : S^1 \times \widehat{W} \rightarrow \mathbb{R}$ is said to be *admissible* if it satisfies the following conditions:

- (i) H is negative and C^2 -small on W ;
- (ii) $H = h(r)$ and $h''(r) \geq 0$ on the cylindrical part $M \times [1, \infty)$;
- (iii) there exists $r_0 \geq 1$ such that $h(r) = kr + c$ for $r \geq r_0$, where $c < 0$ and $k \notin \mathcal{S}_{\widehat{\omega}}(W)$.

If a Hamiltonian satisfies only the condition (iii) then the Hamiltonian is said to be *admissible at infinity*.

Definition 3.1.18. A map $J : S^1 \rightarrow \text{End}(\widehat{W})$ is called an *admissible almost complex structure* if it satisfies the following conditions: for all $t \in S^1$

- (i) $J_t^2 = -I$, where $J_t := J(t)$;
- (ii) J_t is compatible with $\widehat{\omega}$, i.e., $\widehat{\omega}(\cdot, \cdot) = \widehat{\omega}(J_t \cdot, J_t \cdot)$ and $\widehat{\omega}(\cdot, J_t \cdot)$ is a Riemannian metric;
- (iii) $J_t|_\xi = J_0$ and $J_t(r\partial_r) = R_\alpha$,

where R_α is the Reeb vector field of α and J_0 is a compatible complex structure on the symplectic bundle $(\xi, d\alpha)$. Denote by \mathcal{J} the set of admissible almost complex structures.

Remark 3.1.19. Let X_H be the Hamiltonian vector field. By the definition of X_H , we have $(rd\alpha + dr \wedge \alpha)(X_H, \partial_r) = -h'(r)$ on $M \times [1, \infty)$. By computation, we have $X_H = h'(r)R_\alpha$. Thus, the Hamiltonian flow of H on $M \times [1, \infty)$ corresponds to the Reeb flow on M . By the condition (i) in 3.1.17, a closed orbit of the Hamiltonian flow in W is a critical point of H . We refer to [3, Section 6.1] for more details. If x is a Hamiltonian orbit in W , there exists a sufficiently small $\epsilon > 0$ such that for all of the constant orbits, the action

$$\mathcal{A}_H(x) < \epsilon.$$

Let us define the symplectic homology of W . Consider the set of admissible Hamiltonians with a partial order " \leq " defined by

$$H \leq K \text{ if } H \leq K \text{ as functions on } \widehat{W}.$$

Denote by \mathcal{H} a cofinal subset of the partially ordered set. There exists a well-defined continuation map between the Floer homology groups for H and K with $H \leq K$;

$$\Phi : \text{HF}(H) \rightarrow \text{HF}(K),$$

We can take the direct limit of the groups over \mathcal{H} . Define the *symplectic homology* of W by

$$\text{SH}(W) := \varinjlim_{H \in \mathcal{H}} \text{HF}(H).$$

Remark 3.1.20. When we take the direct limit over a cofinal subset \mathcal{H} , we see that $r_0 \rightarrow 1$ and $k \rightarrow \infty$ for r_0 and k in Definition 3.1.17 as well as $H|_W \rightarrow 0$. Thus

for an orbit $x \in \mathcal{P}(H)$ in $M \times [1, r_0)$, the Hamiltonian action

$$\mathcal{A}_H(x) = \int_{\bar{x}} \widehat{\omega} - \int_{S^1} H(t, x(t)) dt = \int_{\bar{x}} \widehat{\omega} - h(r_x),$$

where \bar{x} is a capping of x and the orbit x appears on the level $r = r_x$. As $r_0 \rightarrow 1$, the orbit x approaches a Reeb orbit $\gamma \in \mathcal{P}(\alpha)$; e.g., see Remark 3.1.19 and

$$\mathcal{A}_H(x) \rightarrow \int_{\bar{\gamma}} \widehat{\omega},$$

where $\bar{\gamma}$ is a capping of γ .

Similarly, we define the filtered symplectic homology. Let $I = [a, b] \in \mathbb{R}$ be an interval with the end points outside $\mathcal{S}_{\widehat{\omega}}(W)$. The number a or b can be infinity. Let $\mathrm{HF}^I(H)$ be the Floer homology filtered by the Hamiltonian action \mathcal{A}_H . Then the *filtered symplectic homology* is defined by

$$\mathrm{SH}^I(W) := \varinjlim_{H \in \mathcal{H}} \mathrm{HF}^I(H),$$

where \mathcal{H} is the cofinal subset of admissible Hamiltonians. When $I = (-\infty, \infty)$, $\mathrm{SH}^I(W) = \mathrm{SH}(W)$.

3.1.4 Negative and positive symplectic homology

Assume that α on M is nondegenerate. Let H be a non-positive C^2 -small Morse function on W and a monotone increasing, convex function h of r on $M \times [1, \infty)$ such that $h'' > 0$ in the region containing one-periodic orbits of H . Clearly, H is a Morse-Bott nondegenerate Hamiltonian and such H form a cofinal family. Let $\mathrm{CF}^-(H)$ be the subspace of the Morse-Bott Floer complex $\mathrm{CF}(H)$ of H generated by the critical points of H ; see, e.g., [5]. Actually, $\mathrm{CF}^-(H)$ is a

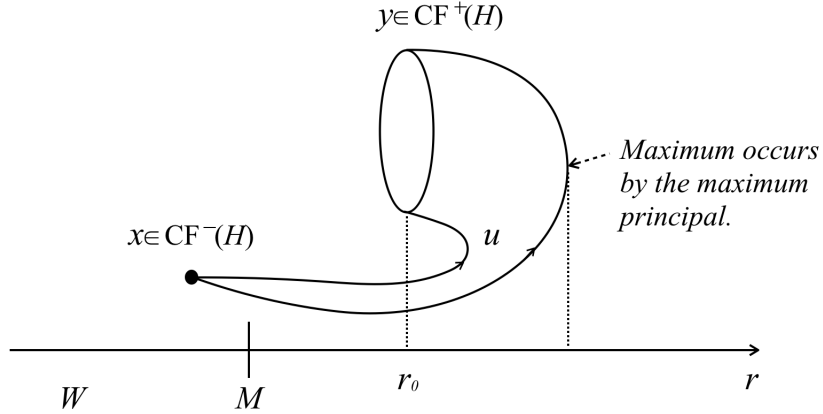


Figure 3.1: There is no Floer trajectory from orbits on $M \times [1, \infty)$ to critical points of H .

subcomplex of $\text{CF}(H)$; see [10, Remark 2]. The reason is that, as shown on [6, p. 654] (see also [14, Lemma 2.3]), a Floer trajectory asymptotic at $+\infty$ to a closed orbit on a level $r = r_0$ can not stay entirely in the union $W \cup_M M \times [1, r_0]$. Thus by the standard maximum principle, such a trajectory cannot be asymptotic to a critical point of H in W at $-\infty$. See Figure 3.1. Hence, $\text{CF}^-(H)$ is closed under the Floer differential. Now we can think of

$$\text{CF}^+(H) = \text{CF}(H) / \text{CF}^-(H),$$

as a Morse–Bott type Floer complex generated by the non-trivial one-periodic orbits of H . The resulting homology groups form a long exact sequence. By passing to the limit over H , we obtain the *negative/positive* symplectic homology groups $\text{SH}^\pm(W)$ which fit into the long exact sequence

$$\cdots \rightarrow \text{SH}^-(W) \rightarrow \text{SH}(W) \rightarrow \text{SH}^+(W) \rightarrow \cdots . \quad (3.1.1)$$

When α is degenerate, we approximate it by nondegenerate forms α' , which

results in a small perturbation W' of W in \widehat{W} , and pass to the the limit as $\alpha' \rightarrow \alpha$. It is easy to see that the resulting negative/positive homology is well defined and we still have the long exact sequence (3.1.1).

The homology $\mathrm{SH}^\pm(H)$ inherits the action filtration in the obvious way and thus we have the groups $\mathrm{SH}^{\pm,I}(H)$, where the end points of I are required to be outside $\mathcal{S}_\omega(H)$. As a consequence, we obtain the action filtration on $\mathrm{SH}^\pm(W)$ with the end points of I outside $\mathcal{S}_\omega(H) \cup \{0\}$. Occasionally, we will use the notation $\mathrm{SH}^{\pm,(-\infty,0]}(W)$ (respectively, $\mathrm{SH}^{(-\infty,0]}(W)$) for the inverse limit of the groups $\mathrm{SH}^{\pm,(-\infty,\epsilon]}(W)$ (respectively, $\mathrm{SH}^{(-\infty,\epsilon]}(W)$) as $\epsilon \searrow 0$.

The long exact sequence (3.1.1) still holds for the filtered groups $\mathrm{SH}^{\pm,I}(H)$ and $\mathrm{SH}^I(W)$. Since the constant orbits of H have non-positive action, $\mathrm{SH}^{-,I}(W) = 0$ if $I \subset (0, \infty)$ and there is a natural map

$$\mathrm{SH}^-(W) \rightarrow \mathrm{SH}^{(-\infty,0]}(W).$$

When the form $\widehat{\omega}$ is exact, this map is an isomorphism.

3.2 S^1 -equivariant symplectic homology

Throughout this section, we assume that (W, ω) is a strong symplectic filling of a contact manifold (M, ξ) such that $\xi = \ker \alpha$ for some 1-form α , and $c_1(TW)|_{\pi_2(W)} = 0$. Let \widehat{W} be the symplectic completion of W .

Consider a *parametrized Hamiltonian*

$$\tilde{H} : S^1 \times \widehat{W} \times S^{2m+1} \rightarrow \mathbb{R}$$

which is invariant with respect to the diagonal S^1 -action, i.e., $\tilde{H}(\theta + t, w, \theta \cdot z) =$

$\tilde{H}(t, w, z)$, where (t, w, z) is a triple of coordinates for $S^1 \times \widehat{W} \times S^{2m+1}$ and the dot $\theta \cdot z$ is the Hopf action of S^1 on S^{2m+1} . Assume that \tilde{H} is admissible at infinity, i.e., there exists $r_0 \geq 1$ such that $\tilde{H}(t, w, z) = kr + c(z)$ for $r \geq r_0$, where $k \notin \mathcal{S}_{\widehat{\omega}}(W)$ and $c(z) = c(\theta \cdot z)$.

Let Λ be the space of contractible loops. Define the *parametrized action functional* on $\Lambda \times S^{2m+1}$ by

$$\mathcal{A}_{\tilde{H}}(x, z) = \int_{\bar{x}} \widehat{\omega} - \int_{S^1} \tilde{H}(t, x(t), z) dt,$$

where \bar{x} is a capping of x . The differential of $\mathcal{A}_{\tilde{H}}$ is

$$(d\mathcal{A}_{\tilde{H}})_{(x,z)}(Y, Z) = \int_{S^1} \widehat{\omega}(Y(t), \dot{x}(t) - X_{\tilde{H}_z}(x(t))) dt + \int_{S^1} \nabla_z \tilde{H}_z(t, x(t)) dt \cdot Z$$

for $Y \in T_x\Lambda$ and $Z \in T_zS^{2m+1}$, where $\tilde{H}_z(t, w) = \tilde{H}(t, w, z)$. Hence, (x, z) is a critical point of $\mathcal{A}_{\tilde{H}}$ if and only if

$$x \in \mathcal{P}(\tilde{H}_z) \quad \text{and} \quad \int_{S^1} \nabla_z \tilde{H}_z(t, x(t)) dt = 0.$$

Denote by $\mathcal{P}(\tilde{H})$ the set of all critical points of $\mathcal{A}_{\tilde{H}}$. The set $\mathcal{P}(\tilde{H})$ is S^1 -invariant, i.e., if (x, z) is a critical point then $\theta \cdot (x(t), z)$ is also a critical point for $\theta \in S^1$, where $\theta \cdot (x(t), z) = (x(t - \theta), \theta \cdot z)$. Denote by $S_{(x,z)}$ the S^1 -orbits of (x, z) . The orbit $S_{(x,z)}$ is called a *critical orbit* of \tilde{H} .

Definition 3.2.1. A critical orbit $S_{(x,z)}$ is said to be *nondegenerate* if the Hessian $d^2\mathcal{A}_{\tilde{H}}$ has a one-dimensional kernel at any point of $S_{(x,z)}$.

Denote by \mathcal{H}^{S^1} the set of parametrized nondegenerate Hamiltonians which are admissible at infinity and all of whose critical orbits are nondegenerate.

Proposition 3.2.2. \mathcal{H}^{S^1} forms a Baire second category in the space of all parametrized

Hamiltonians admissible at infinity. Moreover, if $\tilde{H} \in \mathcal{H}^{S^1}$ then every critical orbit of \tilde{H} is isolated.

For the proof of this proposition, we refer to [7, Proposition 5.1]. By this proposition, there are finitely many critical orbits of $\tilde{H} \in \mathcal{H}^{S^1}$. From now on, we only consider Hamiltonians in \mathcal{H}^{S^1} .

Consider a family \mathcal{J} of (t, z) -dependent almost complex structures J_z^t , for $t \in S^1$ and $z \in S^{2m+1}$, satisfying the following conditions:

- (i) $J_{\theta \cdot z}^{t+\theta} = J_z^t$ for all $\theta \in S^1$;
- (ii) J_z^t is compatible with $\hat{\omega}$;
- (iii) $J_z^t|_{\xi} = J_0$ and $J_z^t \partial_r = R_\alpha$,

where R_α is the Reeb vector field of α and J_0 is a compatible complex structure on the symplectic bundle $(\xi, d\alpha)$. Let us call such an almost complex structure an *S^1 -invariant admissible almost complex structure*. Given $(x, z) \in \Lambda \times S^{2m+1}$, define the metric $\langle \cdot, \cdot \rangle_z$ on $T_x \Lambda \times T_x \Lambda$ by

$$\langle Y, Y' \rangle_z := \int_{S^1} \hat{\omega}(Y(t), J_t^z Y'(t)) dt$$

for $Y, Y' \in T_x \Lambda$. Fix an S^1 -invariant metric on S^{2m+1} . These two metrics give rise to an S^1 -invariant metric g on $\Lambda \times S^{2m+1}$.

Let $p := (x, z)$ and $p' := (x', z')$ be critical points of $\mathcal{A}_{\tilde{H}}$. Given the pair (\mathcal{J}, g) , denote by $\widehat{\mathcal{M}}(S_p, S_{p'}; \tilde{H}, \mathcal{J}, g)$ the space consisting of pairs of functions (u, λ) satisfying the equations:

$$\partial_s u + J_{\lambda(s)}^t \partial_t u - \nabla_w \tilde{H}_{\lambda(s)} = 0,$$

$$\dot{\lambda}(s) - \int_{S^1} \nabla_z \tilde{H}_{\lambda(s)}(t, u(s, t)) dt = 0,$$

$$\lim_{s \rightarrow -\infty} (u(s, t), \lambda(s)) \in S_p \quad \text{and} \quad \lim_{s \rightarrow +\infty} (u(s, t), \lambda(s)) \in S_{p'}, \quad (3.2.1)$$

where $u : \mathbb{R} \times S^1 \rightarrow \widehat{W}$ and $\lambda : \mathbb{R} \rightarrow S^{2m+1}$, and w and z stand for coordinates of \widehat{W} and S^{2m+1} , respectively. Denote the moduli space by

$$\mathcal{M}(S_p, S_{p'}; \tilde{H}, \mathcal{J}, g) := \widehat{\mathcal{M}}(S_p, S_{p'}; \tilde{H}, \mathcal{J}, g)/\mathbb{R}. \quad (3.2.2)$$

For a Hamiltonian $H \in \mathcal{H}^{S^1}$, we can choose an S^1 -invariant metric g on $\Lambda \times S^{2m+1}$ and a family \mathcal{J} of S^1 -invariant admissible almost complex structures such that the moduli space $\mathcal{M}(S_p, S_{p'}; \tilde{H}, \mathcal{J}, g)$ is a smooth manifold of dimension $\mu(p) - \mu(p')$, where $\mu(p)$ is the *parametrized Robbin-Salamon index* of p . We refer to [8, Section 1.3] for the definition of the parametrized Robbin-Salamon index. This moduli space carries a free S^1 -action. Denote by $\mathcal{J}_{reg}(\tilde{H})$ the set of pairs (J, g) such that $J \in \mathcal{J}$, $g(\cdot, \cdot) = \widehat{\omega}(\cdot, J\cdot)$ and the moduli space $\mathcal{M}(S_p, S_{p'}; \tilde{H}, \mathcal{J}, g)$ is a smooth manifold.

Denote the quotient by

$$\mathcal{M}_{S^1}(S_p, S_{p'}; \tilde{H}, \mathcal{J}, g) := \mathcal{M}(S_p, S_{p'}; \tilde{H}, \mathcal{J}, g)/S^1.$$

Then $\mathcal{M}_{S^1}(S_p, S_{p'}; \tilde{H}, \mathcal{J}, g)$ is a smooth manifold of dimension $\mu(p) - \mu(p') - 1$. We refer to [10] for more details.

Remark 3.2.3. As we have seen from the equation (3.2.1), the asymptotes of the moduli space $\mathcal{M}(S_p, S_{p'}; \tilde{H}, \mathcal{J}, g)$ are not fixed. But the critical orbits S_p and $S_{p'}$ are fixed. Since the action functional $\mathcal{A}_{\tilde{H}}$ and the pair (\mathcal{J}, g) are S^1 -invariant, the asymptotes of the quotient $\mathcal{M}_{S^1}(S_p, S_{p'}; \tilde{H}, \mathcal{J}, g)$ are fixed.

We use $\mathcal{M}_{S^1}(S_p, S_{p'})$ as the abbreviation of $\mathcal{M}_{S^1}(S_p, S_{p'}; \tilde{H}, \mathcal{J}, g)$. Given a pair $(J, g) \in \mathcal{J}_{reg}(\tilde{H})$, define the S^1 -equivariant chain complex $(\text{CF}^{S^1, m}(\tilde{H}, J, g), \partial^{S^1, m})$

as

$$\mathrm{CF}_*^{S^1, m}(\tilde{H}, J, g) := \bigoplus_{S_p \subset \mathcal{P}(\tilde{H})} \mathbb{Z}\langle S_p \rangle,$$

where the grading is defined by $\mu(p) + m$. The differential $\partial^{S^1, m}$ is defined by

$$\partial^{S^1, m}(S_p) = \sum_{\substack{S_{p'} \subset \mathcal{P}(\tilde{H}) \\ \mu(p) - \mu(p') = 1}} \langle S_p, S_{p'} \rangle S_{p'}, \quad (3.2.3)$$

where $\langle S_p, S_{p'} \rangle$ is obtained by adding up all the signs of points in the moduli space $\mathcal{M}_{S^1}(S_p, S_{p'})$ and the sign of a point in $\mathcal{M}_{S^1}(S_p, S_{p'})$ is determined by the coherent orientation of $\mathcal{M}_{S^1}(S_p, S_{p'})$. We refer to [10, Proposition 2.2] for the proof of $\partial^{S^1, m} \circ \partial^{S^1, m} = 0$.

For a pair $(J, g) \in \mathcal{J}_{reg}(\tilde{H})$, we define the S^1 -equivariant Floer homology of \tilde{H} as

$$\mathrm{HF}_*^{S^1, m}(\tilde{H}, J, g) := H_*(\mathrm{CF}_*^{S^1, m}(\tilde{H}, J, g), \partial^{S^1, m}).$$

Proposition 3.2.4. *Given two pairs $(J, g), (J', g') \in \mathcal{J}_{reg}(\tilde{H})$, there exists a canonical isomorphism*

$$\mathrm{HF}_*^{S^1, m}(\tilde{H}, J, g) \rightarrow \mathrm{HF}_*^{S^1, m}(\tilde{H}, J', g').$$

We refer to [9, Section 5.3] for the proof. Thus, we do not need to worry about the choice of pairs $(J, g) \in \mathcal{J}_{reg}(\tilde{H})$. Set $\mathrm{HF}_*^{S^1, m}(\tilde{H}) := \mathrm{HF}_*^{S^1, m}(\tilde{H}, J, g)$ for any $(J, g) \in \mathcal{J}_{reg}(\tilde{H})$.

Let \tilde{H}, \tilde{K} be Hamiltonians defined on $\widehat{W} \times S^{2m+1}$ and admissible at infinity. Assume that $H \leq K$. Then there exists a canonical morphism

$$\Phi : \mathrm{HF}_*^{S^1, m}(\tilde{H}) \rightarrow \mathrm{HF}_*^{S^1, m}(\tilde{K}).$$

By considering a cofinal subset of Hamiltonians admissible at infinity, define the homology

$$\mathrm{SH}_*^{S^1, m}(W) := \varinjlim_{\tilde{H}} \mathrm{HF}_*^{S^1, m}(\tilde{H}).$$

Then the S^1 -equivariant symplectic homology is defined by

$$\mathrm{SH}_*^{S^1}(W) := \varinjlim_m \mathrm{SH}_*^{S^1, m}(W),$$

where the direct limit is taken with respect to the embedding $S^{2m+1} \hookrightarrow S^{2m+3}$ that induces the map $\mathrm{SH}_*^{S^1, m}(W) \rightarrow \mathrm{SH}_*^{S^1, m+1}(W)$.

Remark 3.2.5. (Filtered S^1 -equivariant Floer homology)

Let $H : S^1 \times \widehat{W} \rightarrow \mathbb{R}$ be a Hamiltonian admissible at infinity and $I = [a, b] \in \mathbb{R}$ be an interval with end points outside $\mathcal{S}(H)$. One can consider H as a parametrized Hamiltonians independent of t and z , where t and z are coordinates for S^1 and S^{2m+1} , respectively. Define the *filtered S^1 -equivariant Floer homology* of H as

$$\mathrm{HF}_*^{I, S^1}(H) = \varinjlim_m \varinjlim_{\{\tilde{H}\}} \mathrm{HF}_*^{I, S^1, m}(\tilde{H}), \quad (3.2.4)$$

where $\{\tilde{H}\}$ is a cofinal sequence of parametrized Hamiltonians which are transversely nondegenerate perturbations sufficiently close to H . Also, the sequence is converging to H .

Remark 3.2.6. Similar to the definition of the subcomplex $\mathrm{CF}^-(H)$ in Section 3.1.4, the groups

$$\mathrm{CF}_*^{-, S^1, m}(\tilde{H}, J, g) := \bigoplus_{\substack{S_p \subset \mathcal{P}(\tilde{H}) \\ x=\text{constant}}} \mathbb{Z}\langle S_p \rangle$$

build a subcomplex of $(\mathrm{CF}_*^{S^1, m}(\tilde{H}, J, g), \partial^{S^1})$. Denote

$$\mathrm{CF}_*^{+, S^1, m}(\tilde{H}, J, g) := \mathrm{CF}_*^{S^1, m}(\tilde{H}, J, g) / \mathrm{CF}_*^{-, S^1, m}(\tilde{H}, J, g).$$

By taking the direct limit over a cofinal subset of Hamiltonians \tilde{H} and m of the homology groups induced by the above subcomplexes, we obtain, $\mathrm{SH}_*^{-, S^1}(W)$ and $\mathrm{SH}_*^{+, S^1}(W)$.

Lemma 3.2.7. *[10, Lemma 2.3] There exists a natural isomorphism*

$$\mathrm{SH}_*^{-, S^1}(W) \cong \mathrm{H}_{*+n}(W, \partial W) \otimes \mathrm{H}_*(\mathbb{C}P^\infty),$$

where $\dim W = 2n$.

Theorem 3.2.8. *[10, Theorem 1.3]*

(i) *The symplectic homology groups fit into an exact sequence of Gysin type:*

$$\longrightarrow \mathrm{SH}_*^\star(W) \longrightarrow \mathrm{SH}_*^{\star, G}(W) \xrightarrow{D} \mathrm{SH}_{*-2}^{\star, G}(W) \longrightarrow \mathrm{SH}_{*-1}^\star(W) \longrightarrow$$

where $\star = \pm$ or nothing.

(ii) *There is a Leray-Serre type spectral sequence starting with*

$$E_{p, q}^{\star, 2} = \mathrm{SH}_p^\star(W) \otimes \mathrm{H}_q(\mathbb{C}P^\infty)$$

converging to $\mathrm{SH}^{\star, S^1}(W)$, where $\star = \pm$ or nothing.

3.3 Lusternik-Schnirelmann theory

In this section we develop the Lusternik-Schnirelmann theory for the shift operator in S^1 -equivariant symplectic homology.

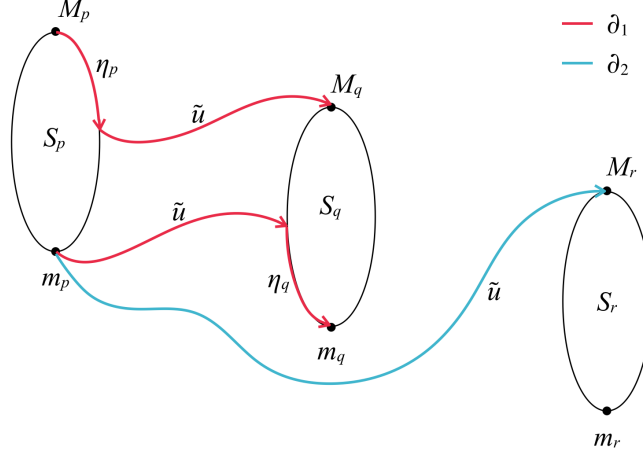


Figure 3.2: The differential ∂ decomposes as $\partial = \partial_1 + \partial_2$.

3.3.1 The shift operator in S^1 -equivariant Floer homology

Under the same setting in Section 3.2, we briefly define the shift operator in S^1 -Floer homology by following [25, Section 2.2.3].

Let S_p be a critical orbit of \tilde{H} , where p is a critical point of the parametrized action functional $\mathcal{A}_{\tilde{H}}$. Fix a Morse function $g_p : S_p \rightarrow \mathbb{R}$ with one maximum M_p and one minimum m_p . Consider the vector space $\text{CF}_*(\tilde{H})$ generated by M_p and m_p and graded by setting $\mu(M_p) = \mu(p) + 1$ and $\mu(m_p) = \mu(p)$, where $\mu(p)$ is the parametrized Robbin-Salamon index of p .

The differential $\partial : \text{CF}_*(\tilde{H}) \rightarrow \text{CF}_{*-1}(\tilde{H})$ decomposes as $\partial = \partial_1 + \partial_2$, where ∂_1 and ∂_2 are defined below. Let p , q , and r be critical points of $\mathcal{A}_{\tilde{H}}$.

When $\mu(q) = \mu(p) - 1$, the first term ∂_1 is defined by

$$\partial_1(M_p) = \sum_q \langle M_p, M_q \rangle M_q \quad \text{and} \quad \partial_1(m_p) = \sum_q \langle m_p, m_q \rangle m_q.$$

Here, $\langle M_p, M_q \rangle$ is obtained by adding up all the signs of points in the moduli space of broken trajectories made of an anti-trajectory $\eta_p : (-\infty, 0] \rightarrow S_p$ of g_p

starting at M_p and the Floer trajectory $\tilde{u}(s, t) = (u(s, t), \lambda(s))$ of (3.2.1) such that the line $\tilde{u}(s, 0)$ connects $\eta(0)$ and m_q . The sign of a point in the moduli space is determined by the coherent orientations. Similarly, $\langle m_p, m_q \rangle$ is obtained by adding up all the signs of points in the moduli space of broken trajectories made of the Floer trajectory from S_p to S_q with $\lim_{s \rightarrow -\infty} \tilde{u}(s, t) = m_p$ and an anti-trajectory η_q of g_q connecting $\lim_{s \rightarrow \infty} \tilde{u}(s, t)$ and m_q . The red line in Figure 3.2 represents the broken trajectories.

When $\mu(r) = \mu(p) - 2$, the second term ∂_2 is defined by

$$\partial_2(m_p) = \sum_r \langle m_p, M_r \rangle M_r \quad \text{and} \quad \partial_2(M_p) = 0,$$

where $\langle m_p, M_r \rangle$ is the number of Floer trajectories from S_p to S_r with the line $\tilde{u}(s, 0)$ connecting m_p and M_r . The green line in Figure 3.2 represents the Floer trajectories.

Then $\partial^2 = 0$. We refer to [9] for the proof. In addition, the homology of the chain complex $(\text{CF}(\tilde{H}), \partial)$ is the ordinary Floer homology of \tilde{H} . Denote the Floer homology by $\text{HF}^m(\tilde{H})$.

Let $C^M(\tilde{H})$ and $C^m(\tilde{H})$ be the vector subspaces generated by $\{M_p \mid p \in \text{Crit}(\mathcal{A}_{\tilde{H}})\}$ and $\{m_p \mid p \in \text{Crit}(\mathcal{A}_{\tilde{H}})\}$, respectively. Then the pair $(C^M(\tilde{H}), \partial_1)$ is a subcomplex of $(\text{CF}(\tilde{H}), \partial)$ and the quotient space $\text{CF}(\tilde{H})/C^M(\tilde{H})$ is isomorphic to $C^m(\tilde{H})$. The pair $(C^m(\tilde{H}), \partial_1)$ is also a subcomplex. Thus, we have the short exact sequence

$$0 \rightarrow (C^M(\tilde{H}), \partial_1) \rightarrow (\text{CF}(\tilde{H}), \partial) \rightarrow (C^m(\tilde{H}), \partial_1) \rightarrow 0 \quad (3.3.1)$$

By recalling the definition of the differential $\partial^{S^1, m}$ in (3.2.3), one can see that

$$H_*(C^m(\tilde{H}), \partial_1) = \mathrm{HF}_{*-1}^{S^1, m}(\tilde{H}) \quad \text{and} \quad H_*(C^m(\tilde{H}), \partial_1) = \mathrm{HF}_*^{S^1, m}(\tilde{H}).$$

The short exact sequence (3.3.1) gives rise to a long exact sequence in homology

$$\cdots \rightarrow \mathrm{HF}_*^m(\tilde{H}) \rightarrow \mathrm{HF}_*^{S^1, m}(\tilde{H}) \xrightarrow{D} \mathrm{HF}_{*-2}^{S^1, m}(\tilde{H}) \rightarrow \mathrm{HF}_{*-1}^m(\tilde{H}) \rightarrow \cdots \quad (3.3.2)$$

The connecting map D is induced by the map $\partial_2 : C^m(\tilde{H}) \rightarrow C^M(\tilde{H})$. We call the map D the *shift operator*.

Let $I = [a, b] \in \mathbb{R}$ be an interval with end points outside $\mathcal{S}(\tilde{H})$. From the exact sequence (3.3.2), we have the same type of exact sequence in homology filtered by the action $\mathcal{A}_{\tilde{H}}$.

$$\cdots \rightarrow \mathrm{HF}_*^{I, m}(\tilde{H}) \rightarrow \mathrm{HF}_*^{I, S^1, m}(\tilde{H}) \xrightarrow{D} \mathrm{HF}_{*-2}^{I, S^1, m}(\tilde{H}) \rightarrow \mathrm{HF}_{*-1}^{I, m}(\tilde{H}) \rightarrow \cdots \quad (3.3.3)$$

Let $H : S^1 \times \widehat{W} \rightarrow \mathbb{R}$ be a Hamiltonian admissible at infinity and $I = [a, b] \in \mathbb{R}$ be an interval with end points outside $\mathcal{S}(H)$. Consider a cofinal sequence of parametrized Hamiltonians \tilde{H} converging to H as we did in Remark 3.2.5. By applying direct limits $\tilde{H} \rightarrow H$ and $m \rightarrow \infty$ to the exact sequence (3.3.3), we obtain the long exact sequence in Floer homology of H

$$\cdots \rightarrow \mathrm{HF}_*^I(H) \rightarrow \mathrm{HF}_*^{I, S^1}(H) \xrightarrow{D} \mathrm{HF}_{*-2}^{I, S^1}(H) \rightarrow \mathrm{HF}_{*-1}^I(H) \rightarrow \cdots \quad (3.3.4)$$

3.3.2 Lusternik-Schnirelmann theory for the shift operator

Let $H : S^1 \times \widehat{W} \rightarrow \mathbb{R}$ be a Hamiltonian admissible at infinity and $I = [a, b] \in \mathbb{R}$ be an interval with end points outside $\mathcal{S}(H)$.

Definition 3.3.1. Assume that H is autonomous. a one-periodic orbit y is said to be a *reparametrization* of x if $y(t) = x(t + \theta)$ for some $\theta \in G = S^1$. Two one-periodic orbits are said to be *geometrically distinct* if one of them is not a reparametrization of the other.

Denote by $\mathcal{P}(H)$ the collection of all geometrically distinct contractible one-periodic orbits of H when H is autonomous. Denote by $\mathcal{P}(H, I)$ the set of one-periodic orbits in $\mathcal{P}(H)$ with the actions in the interval I . For a nonzero class $\xi \in \text{HF}_*^{I, S^1}(H)$, define the *spectral invariant* or *action selector* $c_\xi(H)$ by

$$c_\xi(H) = \inf\{b' \in I \setminus \mathcal{S}(H) \mid \xi \in \text{im}(i_{b'})\},$$

where $i_{b'} : \text{HF}_*^{I', S^1}(H) \rightarrow \text{HF}_*^{I, S^1}(H)$ is the natural map for $I' = [a', b'] \subset I$. Action selectors c_ξ are in $\mathcal{S}(H)$ for a nonzero class $\xi \in \text{HF}_*^{I, S^1}(H)$.

Theorem 3.3.2. [25, Theroem 2.12] Assume that all one-periodic orbits in $\mathcal{P}(H, I)$ are isolated and non-constant. Then for any nonzero class $\xi \in \text{HF}_*^{I, S^1}(H)$, we have

$$c_\xi(H) > c_{D(\xi)}(H),$$

where D is the shift operator in the sequence (3.3.4).

Define the *action gap* by

$$\text{gap}(H) := \min_{\substack{x \neq y \\ x, y \in \mathcal{P}(H, I)}} |\mathcal{A}_H(x) - \mathcal{A}_H(y)|.$$

Then $\text{gap}(H) > 0$.

Corollary 3.3.3. *[25, Corollary 2.14]*

$$c_\xi(H) \geq c_{D(\xi)}(H) + \text{gap}(H) > c_{D(\xi)}(H).$$

Chapter 4

Vanishing of symplectic homology

4.1 Vanishing of symplectic homology on aspherical manifolds

In this section we analyze general quantitative and qualitative consequences of vanishing of the symplectic homology, focusing mainly on symplectically aspherical manifolds.

4.1.1 Homology calculations and equivariant capacities

The condition that $\mathrm{SH}(W) = 0$ readily lends for an explicit calculation of the (equivariant) positive symplectic homology, which then can be used to define several variants of the homological symplectic capacities. We start with a calculation of the negative and positive equivariant symplectic homology of W .

Proposition 4.1.1. *Assume that W^{2n} is symplectically aspherical and $\mathrm{SH}(W) = 0$. Then we have the following natural isomorphisms:*

$$(i) \quad \mathrm{SH}^-(W) = \mathrm{H}_*(W, \partial W)[-n] \text{ and } \mathrm{SH}^{-,S^1}(W) = \mathrm{H}_*(W, \partial W) \otimes \mathrm{H}_*(\mathbb{C}\mathbb{P}^\infty)[-n];$$

(ii) $\mathrm{SH}^+(W) = \mathrm{H}_*(W, \partial W)[-n + 1]$ and

$$\mathrm{SH}^{+,S^1}(W) = \mathrm{H}_*(W, \partial W) \otimes \mathrm{H}_*(\mathbb{C}\mathbb{P}^\infty)[-n + 1]; \quad (4.1.1)$$

(iii) combined with the identification (4.1.1), the Gysin sequence shift map

$$\mathrm{SH}_{r+2}^{+,S^1}(W) \xrightarrow{D} \mathrm{SH}_r^{+,S^1}(W)$$

is the identity on the first factor and the map $\mathrm{H}_{q+2}(\mathbb{C}\mathbb{P}^\infty) \rightarrow \mathrm{H}_q(\mathbb{C}\mathbb{P}^\infty)$, on the second given by the pairing with a suitably chosen generator of $\mathrm{H}^2(\mathbb{C}\mathbb{P}^\infty)$.

In particular, D is an isomorphism when $r \geq n + 1$.

Proof. Assertion (i) is an immediate consequence of the definitions and the condition that W is symplectically aspherical; see, e.g., [9, 61]. With our grading conventions (which ultimately result in the same grading as in [9]), we have

$$\begin{aligned} \mathrm{SH}_r^-(W) &= \mathrm{H}_{n+r}(W, \partial W), \\ \mathrm{SH}_r^{-,S^1}(W) &= \bigoplus_{p+q=r} \mathrm{H}_{p+n}(W, \partial W) \otimes \mathrm{H}_q(\mathbb{C}\mathbb{P}^\infty), \end{aligned}$$

where all homology groups are taken with coefficients in \mathbb{Q} . In particular, as W is oriented,

$$\begin{aligned} \mathrm{SH}_n^-(W) &= \mathbb{Q}, \\ \mathrm{SH}_q^-(W) &= 0 \text{ if } q \geq n + 1 \end{aligned}$$

Combining the assumption $\mathrm{SH}(W) = 0$ with the long exact sequence

$$\begin{array}{ccc}
\mathrm{SH}(W) & \longrightarrow & \mathrm{SH}^+(W) \\
& \swarrow & \searrow \\
& & \mathrm{SH}^-(W)
\end{array}
\begin{array}{l}
\\
\\
[-1]
\end{array}$$

we see that $\mathrm{SH}_{q+1}^+(W) = \mathrm{SH}_q^-(W)$. Hence, we have

$$\begin{aligned}
\mathrm{SH}_{n+1}^+(W) &= \mathrm{SH}_n^-(W) = \mathbb{Q}, \\
\mathrm{SH}_q^+(W) &= 0 \text{ if } q \geq n + 2.
\end{aligned} \tag{4.1.2}$$

This proves the second assertion.

Next consider the Gysin sequence

$$\begin{array}{ccc}
\mathrm{SH}_*^{+,S^1}(W) & \xrightarrow{D} & \mathrm{SH}_{*-2}^{+,S^1}(W) \\
& \swarrow & \searrow \\
& & \mathrm{SH}_*^+(W)
\end{array}
\begin{array}{l}
\\
\\
[+1]
\end{array}$$

where D is the shift operator. Then

$$\mathrm{SH}_{q+2}^{+,S^1}(W) \cong \mathrm{SH}_q^{+,S^1}(W) \text{ if } q \geq n + 1.$$

By assertions (i) and (ii) of Theorem 3.2.8, we have that $\mathrm{SH}(W) = 0$ if and only if $\mathrm{SH}^{S^1}(W) = 0$. From the long exact sequence, we see that

$$\mathrm{SH}_{r+1}^{+,S^1}(W) = \mathrm{SH}_r^{-,S^1}(W) = \bigoplus_{p+q=r} \mathrm{H}_{p+n}(W, \partial W) \otimes \mathrm{H}_q(\mathbb{C}P^\infty).$$

This isomorphism commutes with D and, on the right, D is given by the pairing $\mathrm{H}_q(\mathbb{C}P^\infty) \rightarrow \mathrm{H}_{q-2}(\mathbb{C}P^\infty)$ with a generator of $\mathrm{H}^2(\mathbb{C}P^\infty)$. This proves assertion

(iii) and completes the proof of the theorem. \square

With this calculation in mind, we define homological symplectic capacities, a.k.a. spectral invariants, a.k.a. action selectors, depending on the perspective. The construction follows the standard path which goes back to [15, 34, 55, 60]. (See also [25, 33] for a recent detailed treatment in the case where W is a ball.)

To a non-zero class $\beta \in \text{SH}^+(W)$, we associate the *capacity*

$$c(\beta, W) = \inf\{a \in \mathbb{R} \mid \beta \in \text{im}(i_a)\} \in \mathbb{R},$$

where the map $i_a: \text{SH}^{+,-\infty,a}(W) \rightarrow \text{SH}^+(W)$ is induced by the inclusion of the complexes. (When $\beta = 0$, we have, by definition, $c(\beta, W) = -\infty$.) This capacity can be viewed as a function of β or W . In the latter case, $c(\beta, W)$ has all expected features of a symplectic capacity as long as W varies within a suitably chosen class of manifolds with naturally isomorphic homology groups $\text{SH}^+(W)$. For $\beta \in \text{SH}^{+,S^1}(W)$, the equivariant capacity $c^{S^1}(\beta, W)$ is defined in a similar fashion.

By assertion (ii) of Proposition 4.1.1, every class $\zeta \in H_*(W, \partial W)$ gives rise to class $\zeta^+ \in \text{SH}^+(W)$ and we set $c_\zeta(W) = c(\zeta^+, W)$. The capacity arising from the unit $\zeta = [W, \partial W]$ is of particular interest and we denote it by $c(W)$. Likewise, by (4.1.1), we can associate to ζ a sequence of classes $\zeta_k^{S^1} = \zeta^+ \otimes \sigma_k \in \text{SH}^{+,S^1}(W)$ for $k = 0, 1, 2, \dots$, where σ_k is a generator in $H_{2k}(\mathbb{C}\mathbb{P}^\infty)$ and $D(\zeta_{k+1}^{S^1}) = \zeta_k^{S^1}$. We set

$$c_{\zeta,k}^{S^1}(W) := c(\zeta_k^{S^1}, W).$$

When $\zeta = [W, \partial W]$, we write $c_k^{S^1} := c_{\zeta,k}^{S^1}$. The operator D does not increase the

action filtration (see, e.g., [9, 25]). Hence,

$$c_{\zeta,0}^{S^1}(W) \leq c_{\zeta,1}^{S^1}(W) \leq c_{\zeta,2}^{S^1}(W) \leq \dots \quad (4.1.3)$$

Lemma 4.1.2. *The capacities are non-negative:*

$$c_{\zeta}(W) \geq 0 \text{ and } c_{\zeta,k}^{S^1}(W) \geq 0 \quad (4.1.4)$$

and

$$c_{\zeta,0}^{S^1}(W) \leq c_{\zeta}(W). \quad (4.1.5)$$

These inequalities are well known when W is exact. Moreover, then all capacities are strictly positive. However, when W is only assumed to be symplectically aspherical, non-trivial closed Reeb orbits on ∂W can possibly have negative Hamiltonian action, and (4.1.4) is not entirely obvious.

Proof. To prove (4.1.4) for $c_{\zeta}(W)$, consider an admissible Hamiltonian H which is nondegenerate and bounded below by $-\delta < 0$ on W . It is clear that the action selector corresponding to ζ^+ for H is also bounded from below by $-\delta$. Indeed, after a small nondegenerate perturbation of H outside W , the value of the selector is attained on an orbit which is connected by a Floer trajectory to a critical point of H in W . Passing to the limit, we see that $c_{\zeta}(W) \geq 0$. For the capacities $c_{\zeta,k}^{S^1}$ the argument is similar.

The proof of (4.1.5) is identical to the argument in the case where W is exact. Namely, by arguing as in the proof of Proposition 4.1.1 it is easy to show that the natural map

$$H_{n+*}(W, \partial W) \xrightarrow{\cong} SH_{*}^{-}(W) \xrightarrow{\cong} SH_{*+1}^{+}(W) \rightarrow SH_{*+1}^{+,S^1}(W),$$

where we suppressed in the notation the grading shift by the second arrow isomorphism, sends ζ to $\zeta_0^{S^1} = \zeta^+ \otimes \sigma_0$. With this in mind, (4.1.5) follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{SH}^{+,(-\infty,a)}(W) & \longrightarrow & \mathrm{SH}^+(W) \\ \downarrow & & \downarrow \\ \mathrm{SH}^{+,(-\infty,a),S^1}(W) & \longrightarrow & \mathrm{SH}^{+,S^1}(W) \end{array}$$

□

Remark 4.1.3. We expect that the strict inequalities also hold in (4.1.4). However, proving this requires a more subtle argument. One can use, for instance, a continuation or “transfer” map between W and a slightly shrunk domain W' to show that this map decreases the action by a certain amount and reasoning as in the proof of Theorem 4.1.9. In Section 4.2, we will show that this is true for prequantization bundles by a rather simple and different argument.

4.1.2 Uniform instability of the symplectic homology

Let us now turn to quantitative consequences of vanishing of the symplectic homology.

We say that the filtered symplectic homology of W is *uniformly unstable* if the natural “quotient-inclusion” map

$$\mathrm{SH}^I(W) \rightarrow \mathrm{SH}^{I+c}(W) \tag{4.1.6}$$

is zero for every interval I (possibly infinite) and some constant $c \geq 0$ independent of I . One way to interpret this definition, inspired by the results in [58], is that every element of the filtered homology is “noise” on the c -scale or, equivalently,

all bars in the barcode associate with this homology have length no longer than c . (See, e.g., [47, 62] for a discussion of barcodes and persistence modules in the context of symplectic topology.)

The requirement that the homology is uniformly unstable is seemingly stronger than that the global homology vanishes: setting $I = \mathbb{R}$ we conclude that $\text{SH}(W) = 0$. However, as was pointed out to us by Kei Irie, [37], the two conditions are equivalent for Liouville domains. In other words, somewhat surprisingly, vanishing of the global homology is equivalent to the uniform instability of the filtered homology. The next proposition is a minor generalization of his observation.

Proposition 4.1.4. *Assume that $\omega|_{\pi_2(W)} = 0$. The following two conditions are equivalent:*

- (i) $\text{SH}(W) = 0$ and
- (ii) *there exists a constant $c_0 > 0$ such that for any $c > c_0$ and any interval $I \subset \mathbb{R}$ the map (4.1.6) is zero.*

Moreover, the smallest constant c_0 with this property is exactly the capacity $c(W)$.

Proof. As has been pointed out above, to prove the implication (ii) \Rightarrow (i), it is enough to set $I = \mathbb{R}$ in (ii). Indeed, then (4.1.6) is simultaneously zero and the identity map, which is only possible when $\text{SH}(W) = 0$.

Let us prove the converse. Assume that $\text{SH}(W) = 0$ and consider the natural map $\psi: \text{SH}^-(W) \rightarrow \text{SH}^{(-\infty, c)}(W)$. By definition, $c(W) = \inf\{c \mid \psi(\zeta) = 0\} < \infty$ where we took ζ to be the image of the fundamental class $[W, \partial W] \in \text{SH}^-(W)$; see Proposition 4.1.1. Our goal is to show that the map (4.1.6) vanishes for any $c > c(W)$ and any interval I .

Step 1. For $a \notin \mathcal{S}_{\widehat{\omega}}(W)$, consider an interval $I = (-\infty, a)$. We have the following commutative diagram where the horizontal maps are given by the pair-of-pants product:

$$\begin{array}{ccc} \mathrm{SH}^{(-\infty, a)}(W) \otimes \mathrm{SH}^{(-\infty, 0]}(W) & \longrightarrow & \mathrm{SH}^{(-\infty, a)}(W) \\ \mathrm{id} \otimes \psi \downarrow & & \downarrow \phi \\ \mathrm{SH}^{(-\infty, a)}(W) \otimes \mathrm{SH}^{(-\infty, c)}(W) & \longrightarrow & \mathrm{SH}^{(-\infty, a+c)}(W) \end{array}$$

Recall that $\zeta \in \mathrm{SH}^-(W)$ is a unit with respect to this product. (We refer the reader to [1] for the definition of the pair-of-pants product applicable in this case and also to, e.g., [49].) Thus, for any $\sigma \in \mathrm{SH}^{(-\infty, a)}(W)$,

$$\begin{array}{ccc} \sigma \otimes \zeta & \longmapsto & \sigma \\ \downarrow & & \downarrow \\ \sigma \otimes 0 & \longmapsto & 0. \end{array}$$

Hence, the map ϕ vanishes.

Step 2. For $a, b \notin \mathcal{S}_{\widehat{\omega}}(W)$, consider an interval $I = (a, b)$. We have the the following commutative diagram:

$$\begin{array}{ccccccc} \rightarrow & \mathrm{SH}_k^{(-\infty, a)}(W) & \longrightarrow & \mathrm{SH}_k^{(-\infty, b)}(W) & \longrightarrow & \mathrm{SH}_k^{(a, b)}(W) & \longrightarrow \\ & \downarrow \phi_1 & & \downarrow \phi_2 & & \downarrow \psi & \\ \rightarrow & \mathrm{SH}_k^{(-\infty, a+c)}(W) & \longrightarrow & \mathrm{SH}_k^{(-\infty, b+c)}(W) & \longrightarrow & \mathrm{SH}_k^{(a+c, b+c)}(W) & \longrightarrow \end{array}$$

By Step 1, the maps ϕ_1 , ϕ_2 are zero maps. Hence the map ψ vanishes.

Step 3. For $a \notin \mathcal{S}_{\widehat{\omega}}(W)$ consider an interval $I = (a, \infty)$. At Step 2, we obtained the zero map $\psi: \mathrm{SH}^{(a, b)}(W) \rightarrow \mathrm{SH}^{(a+c, b+c)}(W)$. By taking b to ∞ , we see the map $\psi: \mathrm{SH}^{(a, \infty)}(W) \rightarrow \mathrm{SH}^{(a+c, \infty)}(W)$ vanishes. \square

Remark 4.1.5. It is worth pointing out that Proposition 4.1.4 and Theorem 4.1.9 below do not have a counterpart in the equivariant setting. Indeed, when W is

the standard symplectic ball B^{2n} the maps

$$\mathrm{SH}^{S^1, (a, \infty)}(W) \rightarrow \mathrm{SH}^{S^1, (a+c, \infty)}(W)$$

are non-zero for any a and $c \geq 0$ while $\mathrm{SH}^{S^1}(W) = 0$.

4.1.3 Growth of symplectic capacities

Another consequence of vanishing of the symplectic homology is an upper bound on the growth of the equivariant symplectic capacities.

Proposition 4.1.6. *Assume that $\omega|_{\pi_2(W)} = 0$. Then, for every $\zeta \in H_d(W, \partial W)$ and k such that $2k \geq 2n - d$, we have*

$$0 \leq c_{\zeta, k+1}^{S^1}(W) - c_{\zeta, k}^{S^1}(W) \leq c(W).$$

Proof. The first inequality is simply the assertion that the sequence $c_{\zeta, k}(W)$ is (non-strictly) monotone increasing (see (4.1.3)) and, as has been pointed out in Section 4.1.1, this is a consequence of the fact that the operator D does not increase the action filtration (see, e.g., [9, 25]).

Let us show that $c_{\zeta, k+1}^{S^1}(W) - c_{\zeta, k}^{S^1}(W) \leq c(W)$. By Proposition 4.1.1,

$$\begin{aligned} \mathrm{SH}_{n+1}^{+, S^1}(W) &\cong \bigoplus_{k=0}^n H_{2n-2k}(W, \partial W), \\ \mathrm{SH}_{n+2}^{+, S^1}(W) &\cong \bigoplus_{k=1}^n H_{2n-2k+1}(W, \partial W). \end{aligned}$$

For $r \geq 1$ and $e > c(W)$, we have the following commutative diagram:

$$\begin{array}{ccccccc}
\rightarrow & \mathrm{SH}_{n+r}^{(b+e,\infty)}(W) & \rightarrow & \mathrm{SH}_{n+r+2}^{(b+e,\infty),S^1}(W) & \rightarrow & \mathrm{SH}_{n+r}^{(b+e,\infty),S^1}(W) & \rightarrow \dots \\
& \uparrow (4.1.6) & & \uparrow f & & \uparrow & \\
\rightarrow & \mathrm{SH}_{n+r+2}^{(b,\infty)}(W) & \xrightarrow{\pi_*} & \mathrm{SH}_{n+r+2}^{(b,\infty),S^1}(W) & \xrightarrow{D_1} & \mathrm{SH}_{n+r}^{(b,\infty),S^1}(W) & \rightarrow \dots \\
& \uparrow & & \uparrow j_{r+2} & & \uparrow j_r & \\
\rightarrow & \mathrm{SH}_{n+r+2}^+(W) & \rightarrow & \mathrm{SH}_{n+r+2}^{+,S^1}(W) & \xrightarrow{D_2} & \mathrm{SH}_{n+r}^{+,S^1}(W) & \rightarrow \mathrm{SH}_{n+r+1}^+(W) \rightarrow \\
& \uparrow & & \uparrow i_{r+2} & & \uparrow i_r & \\
\rightarrow & \mathrm{SH}_{n+r+2}^{(\epsilon,b)}(W) & \rightarrow & \mathrm{SH}_{n+r+2}^{(\epsilon,b),S^1}(W) & \rightarrow & \mathrm{SH}_{n+r}^{(\epsilon,b),S^1}(W) & \rightarrow \dots \\
& \uparrow & & \uparrow & & \uparrow &
\end{array}$$

Except for the first row, each row is the Gysin sequence and D_i is the shift operator for $i = 1, 2$; each column comes from the short exact sequence

$$0 \longrightarrow \mathrm{CF}_*^{(\epsilon,b)} \longrightarrow \mathrm{CF}_*^{(\epsilon,\infty)} \longrightarrow \mathrm{CF}_*^{(b,\infty)} \longrightarrow 0,$$

where CF_*^I is a filtered Floer complex; the first row is obtained by shifting the action interval (b, ∞) by e upward.

Consider a class $\zeta \in \mathrm{H}_*(W, \partial W)$. Since $2k \geq 2n - d$, the class $\zeta_k^{S^1}$ lies in $\mathrm{SH}_{n+r}^{+,S^1}(W)$ for some $r \geq 1$. From the fact (4.1.2), we see that $\mathrm{SH}_{n+r+2}^+(W) = 0$ and $\mathrm{SH}_{n+r+1}^+(W) = 0$ for all $r \geq 1$. Hence the map D_2 is an isomorphism. Let ξ be the preimage of $\zeta_k^{S^1}$ under D_2 . Assume that there exists a class $\zeta' \in \mathrm{SH}_{n+r}^{(\epsilon,b),S^1}(W)$ which is sent to $\zeta_k^{S^1}$ by the map i_r . Then we see that $(i_r \circ j_r)(\zeta') = 0$. By commutativity of the diagram, $(D_1 \circ j_{r+2})(\xi) = 0$. Hence, there exists a class $\xi' \in \mathrm{SH}_{n+r+2}^{(b,\infty)}(W)$ such that $\pi_*(\xi') = j_{r+2}(\xi)$. Again, by commutativity of the diagram, $(f \circ j_{r+2})(\xi) = 0$. Hence, $c_{\zeta,k+1}^{S^1}(W) \leq b + e$. \square

4.1.4 Vanishing and displacement

A geometrical counterpart of the condition that $\text{SH}(W) = 0$ is the requirement that W is (stably) displaceable in \widehat{W} . In this section, we will revisit and generalize the well-known fact that $\text{SH}(W) = 0$ for displaceable Liouville domains W . In particular, we extend this result to monotone or negative monotone symplectic manifolds.

Definition 4.1.7. Let (V, ω) be a closed symplectic manifold and A be a subset of V . Denote by $\mathcal{H}_c(S^1 \times V)$ the set of time-dependent Hamiltonians with compact support. When A is compact, the *displacement energy* of A in V is defined as

$$e(A, V) := \inf\{\|H\| \mid H \in \mathcal{H}_c(S^1 \times V), \varphi_H(A) \cap A = \emptyset\},$$

where φ_H is the time-one map of H and the Hofer norm $\|H\|$ is defined as

$$\|H\| := \int_{S^1} \left(\max_{x \in V} H(t, x) - \min_{x \in V} H(t, x) \right) dt.$$

When A is an arbitrary subset, the displacement energy is

$$e(A, V) := \sup\{e(K, V) \mid K \subset V \text{ is compact}\}.$$

The *stable displacement energy* of A in V is defined as

$$e_{st}(A, V) := e(A \times S^1, V \times T^*S^1, \omega \oplus \omega_0),$$

where ω_0 is the standard symplectic form on T^*S^1 . Throughout the rest of this thesis, we abbreviate $e(A, V)$ as $e(A)$ and $e_{st}(A, V)$ as $e_{st}(A)$.

Definition 4.1.8. In the setting of Definition 4.1.7, a compact subset A of V is

said to be *displaceable* in V if there exists a Hamiltonian $H \in \mathcal{H}_c(S^1 \times V)$ such that $\varphi_H(A) \cap A = \emptyset$. Also, we say that a compact set A of V is *stably displaceable* if $A \times S^1$ is displaceable in $V \times T^*S^1$.

The following theorem generalizes the result that $\text{SH}(W) = 0$ for Liouville domains displaceable in \widehat{W} proved in [12, 13] via vanishing of the Rabinowitz Floer homology. (See also [61] for first results in this direction.)

Theorem 4.1.9. *Assume that W is positive or negative monotone and that W is displaceable in \widehat{W} with displacement energy $e(W)$. Then for any $c > e(W)$ and any interval $I \subset \mathbb{R}$ the quotient-inclusion map (4.1.6) is zero. Thus the filtered symplectic homology is uniformly unstable and, in particular, $\text{SH}(W) = 0$.*

The proof of Theorem 4.1.9 when $\omega|_{\pi_2(W)} = 0$ is implicitly contained in [58]. Thus the main new point here is that this condition can be relaxed as that W is allowed to be positive or negative monotone. Note also that when W is symplectically aspherical one can obtain the uniform instability as a consequence of Proposition 4.1.4 and of vanishing of the homology although with a possibly different lower bound on c , which turns out to be better. Namely, combining this proposition with Theorem 4.1.9, and also using Proposition 4.1.6, we have the following:

Corollary 4.1.10. *Assume that W is symplectically aspherical and displaceable in \widehat{W} with displacement energy $e(W)$. Then*

$$c(W) \leq e(W)$$

and thus, for $\zeta \in H_d(W, \partial W)$ and k such that $2k \geq 2n - d$,

$$0 \leq c_{\zeta, k+1}^{S^1}(W) - c_{\zeta, k}^{S^1}(W) \leq e(W).$$

Combining the Künneth formula from [43] with Theorem 4.1.9, we obtain the following well-known result.

Corollary 4.1.11. *Assume that W is Liouville and that W is stably displaceable in \widehat{W} with stable displacement energy $e_{st}(W)$. Then for any $c > e_{st}(W)$ and any interval $I \subset \mathbb{R}$ the map (4.1.6) is zero. In particular, $\text{SH}(W) = 0$.*

Proof of Theorem 4.1.9. First, assume that W is aspherical and $I = [a, b]$ for $a, b \notin \mathcal{S}(\alpha)$. Consider a cofinal sequence of admissible Hamiltonians $\{H_i: S^1 \times \widehat{W} \rightarrow \mathbb{R}\}$ satisfying the following conditions:

- (i) H_i is C^2 -small on W ;
- (ii) $H_i = h_i(r)$ on the cylindrical part $\partial W \times [1, \infty)$, where

$$\begin{aligned} h_i''(r) &\geq 0 && \text{if } r \in [1, r_i] \\ h_i(r) &= k_i r + l_i && \text{if } r \in [r_i, \infty) \end{aligned}$$

for $k_i \notin \mathcal{S}(\alpha)$ and some $r_i > 0$;

- (iii) $k_i \rightarrow \infty$ and $r_i \rightarrow 1$ as $i \rightarrow \infty$

Define a sequence of Hamiltonians $\{F_i: S^1 \times \widehat{W} \rightarrow \mathbb{R}\}$ by

- (i) F_i is on C^2 -small W ;
- (ii) $F_i = f_i(r)$ on the cylindrical part $\partial W \times [1, \infty)$, where for $\epsilon > 0$

$$f_i(r) = \begin{cases} h_i(r) & \text{if } r \in [1, r_i] \\ k_i^-(r - r_i) & \text{if } r \in [r_i, r_i^- - \epsilon] \\ c_i & \text{if } r \in [r_i^-, r_i^+] \\ k_i^+(r - r_i) & \text{if } r \in [r_i^+ + \epsilon, \infty) \end{cases}$$

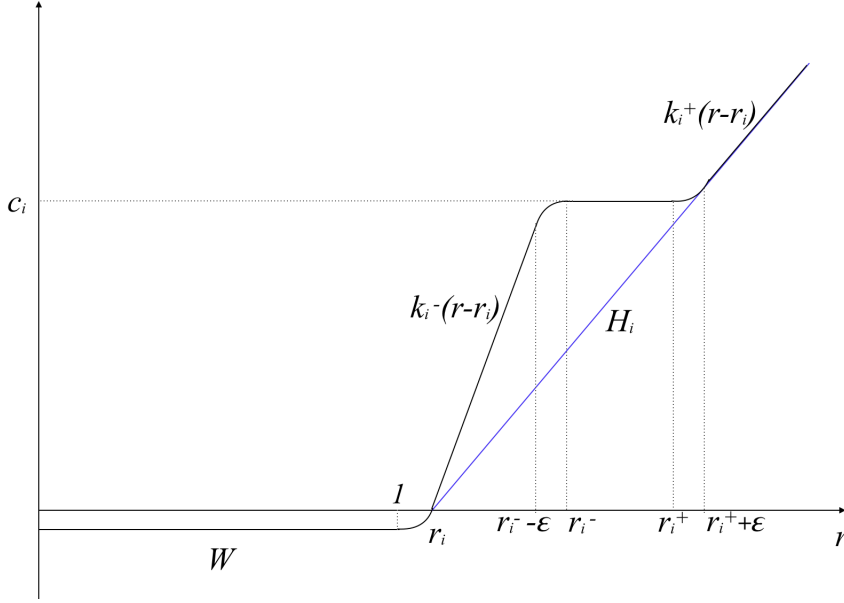


Figure 4.1: The graphs of F_i and H_i

for $k_i^+ = k_i$, $k_i^- \notin \mathcal{S}(\alpha)$ and $k_i^- > k_i^+$;

(iii) $f_i''(r) \leq 0$ if $r \in [r_i^- - \epsilon, r_i^-]$ and $f_i''(r) \geq 0$ if $r \in [r_i^+, r_i^+ + \epsilon]$;

(iv) $\min F_i \rightarrow 0$ as $i \rightarrow \infty$.

The graphs of F_i and H_i are illustrated in Figure 4.1. Then the sequence $\{F_i\}$ is cofinal and each $F_i \geq H_i$. Thus, we have

$$\mathrm{SH}^I(W) = \varinjlim_i \mathrm{HF}^I(F_i),$$

where the limit is taken over the cofinal sequence $\{F_i\}$.

We show that the map $\mathrm{HF}^I(F) \rightarrow \mathrm{HF}^{I+e}(F)$ is zero for a Hamiltonian $F \in \{F_i\}_{i \geq N}$ and sufficiently large N . By the assumption that W is displaceable in \widehat{W} , there exists a Hamiltonian $K: S^1 \times \widehat{W} \rightarrow \mathbb{R}$ such that $\phi_K^1(W) \cap W = \emptyset$ and $e(W) < \|K\| < e$, where ϕ_K^t is the Hamiltonian flow of K . Consider the positive

and negative parts of Hofer's norm of K :

$$\|K\|_+ = \int_{S^1} \max_{x \in \widehat{W}} K(t, x) dt$$

and

$$\|K\|_- = \int_{S^1} -\min_{x \in \widehat{W}} K(t, x) dt.$$

Then $\|K\| = \|K\|_+ + \|K\|_-$. Choose a constant $s \gg 0$ meeting the following conditions:

$$\begin{aligned} \inf \mathcal{S}(H) + s &> b + \|K\|_+, \\ \inf \mathcal{S}(K) + s &> b + \|K\|_+. \end{aligned}$$

Select constants c and r_{\pm} such that

$$\begin{aligned} c &> s, \\ \text{supp } K &\subset W \cup \partial W \times [1, r^+], \\ \phi_K^1 &\text{ displaces } W \cup \partial W \times [1, r^-]. \end{aligned}$$

Define a Hamiltonian $K \# F$ by

$$K \# F(t, x) = K(t, x) + F\left(t, \left(\phi_K^t\right)^{-1}(x)\right).$$

Then $\phi_{K \# F}^t = \phi_K^t \circ \phi_F^t$ which is homotopic to the catenation of ϕ_F^t with ϕ_K^t .

Let $\mathcal{P}(K)$ be the collection of one-periodic orbits of K which are contractible in W . Define the collections $\mathcal{P}(F)$ and $\mathcal{P}(K \# F)$ similarly. Then there is one-to-one correspondence between $\mathcal{P}(H)$ and $\mathcal{P}(F)$ for orbits lying on a level $r \in [r^+, r^+ + \epsilon]$. Denote by $\mathcal{P}(F, r^+)$ the collection of such orbits in $\mathcal{P}(F)$. It is not

hard to see that $\mathcal{P}(K\#F)$ consists of the orbits in $\mathcal{P}(K)$ and the orbits in $\mathcal{P}(F, r^+)$. Indeed, all of the orbits in $\mathcal{P}(F)$ on $r \leq r^-$ are displaced by ϕ_K^1 . Thus, the orbits near $r = r^+$ survive the displacement by ϕ_K^1 .

Evaluating the action functional for $x \in \mathcal{P}(K)$ and $y \in \mathcal{P}(F, r^+)$,

$$\begin{aligned} \mathcal{A}_{K\#F}(x) &= - \int_{\bar{x}} \hat{\omega} + \int_{S^1} K\#F(t, x(t)) dt \\ &= \mathcal{A}_K(x) + \int_{S^1} F\left(t, (\phi_K^t)^{-1}(x(t))\right) dt \\ &= \mathcal{A}_K(x) + c \\ &\geq b + \|K\|_+, \end{aligned}$$

where \bar{x} is a capping of x .

Let $z \in \mathcal{P}(H)$ be the orbit corresponding to $y \in \mathcal{P}(F, r^+)$. Then

$$\begin{aligned} \mathcal{A}_{K\#F}(y) &= - \int_{\bar{y}} \hat{\omega} + \int_{S^1} K\#F(t, y(t)) dt \\ &= - \int_{\bar{z}} \hat{\omega} + (r^+ - 1) \int_z \alpha + \int_{S^1} H(t, z(t)) dt + c \\ &= \mathcal{A}_H(z) + (r^+ - 1) \int_z \alpha + c \\ &\geq b + \|K\|_+, \end{aligned}$$

where \bar{y} and \bar{z} are cappings of y and z , respectively.

Let H_s be a linear homotopy from F to $K\#F$. For $x \in \mathcal{P}(F)$ and $y \in \mathcal{P}(K\#F)$, consider the moduli space

$$\begin{aligned} \mathcal{M}(x, y, H_s, J_s) &= \{u \in C^\infty(S^1 \times \mathbb{R}, \widehat{W}) \mid \lim_{s \rightarrow -\infty} u(t, s) = x, \lim_{s \rightarrow +\infty} u(t, s) = y, \\ &\quad \partial_s u + J_s(\partial_t u - X_{H_s}(u)) = 0\}. \end{aligned}$$

If the moduli space is not empty,

$$\begin{aligned}
\mathcal{A}_{K\#F}(y) &\leq \mathcal{A}_F(x) + \int_{S^1} \int_{\mathbb{R}} \frac{\partial H_s}{\partial s}(u) ds dt \\
&\leq \mathcal{A}_F(x) + \int_{S^1} \max_{x \in \widehat{W}} (K\#F - F) dt \\
&= \mathcal{A}_F(x) + \|K\|_+.
\end{aligned}$$

Similarly, consider a linear homotopy from $K\#F$ to F . Then for $x \in \mathcal{P}(F)$ and $y \in \mathcal{P}(K\#F)$, we have

$$\mathcal{A}_F(x) \leq \mathcal{A}_{K\#F}(y) + \|K\|_-.$$

For $e > \|K\|$, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{HF}^I(F) & \xrightarrow{(1)} & \mathrm{HF}^{I+e}(F) \\
\downarrow (2) & & \uparrow \\
\mathrm{HF}^{I+\|K\|_+}(K\#F) & \longrightarrow & \mathrm{HF}^{I+\|K\|_++\|K\|_-}(F)
\end{array}$$

Since $\mathcal{A}_{K\#F}(x) \geq b + \|K\|_+$ for every $x \in \mathcal{P}(K\#F)$, the map (2) vanishes. The map (1) vanishes as well. By taking direct limit of the map (1) over the cofinal sequence $\{F_i\}$, we see that the map (4.1.6) vanishes.

Next let us show that the map (4.1.6) vanishes when W is monotone, i.e., $[\omega]|_{\pi_2(W)} = \lambda c_1(TW)|_{\pi_2(W)}$ for some nonzero constant λ . Let H be an admissible Hamiltonian. Consider the set

$$\mathcal{S}_q(H) = \{\mathcal{A}_H(\bar{x}) \mid \Delta_H(\bar{x}) \in [q, q + 2n]\},$$

where $\dim W = 2n$. The set $\mathcal{S}_q(K)$ is defined similarly for the Hamiltonian K .

Clearly, these sets are compact. Now, choose a constant $s \gg 0$ meeting the following conditions:

$$\begin{aligned}\inf \mathcal{S}_q(H) + s &> b + \|K\|_+, \\ \inf \mathcal{S}_q(K) + s &> b + \|K\|_+.\end{aligned}$$

By following the same process as the case of that W is aspherical, we conclude that the map (4.1.6) is zero. \square

4.2 Vanishing of symplectic homology for pre-quantization bundles

Throughout this section, we keep the notation and convention from Section 2.3. Furthermore, *all loops and periodic orbits are assumed to be contractible in E unless stated otherwise.*

Recall that to every such loop x in M_α we can associate two actions: the *symplectic area* $\mathcal{A}_\omega(x)$ obtained by integrating ω over a disk bounded by x in E or W and the *contact action* $\mathcal{A}_\alpha(x)$ which is the integral of α over x . These two actions are in general different.

Example 4.2.1. Let M be the S^1 -bundle $r = \epsilon$ in E . Then the closed Reeb orbits x in M are the iterated fibers. (In particular, every Reeb orbit is closed.) As a straightforward calculation shows, we have $\mathcal{A}_\omega(x) = \pi k \epsilon^2 / 2$ and $\mathcal{A}_\alpha(x) = \pi k(1 + \epsilon^2) / 2$ for $x \in \mathfrak{f}^k$.

Lemma 4.2.2. *Let x be a loop in M_α in the free homotopy class \mathfrak{f}^k , $k \in \mathbb{Z}$, i.e., $L_B(x) = k$. Then*

$$\mathcal{A}_\omega(x) = \mathcal{A}_\alpha(x) - \frac{\pi}{2}k. \tag{4.2.1}$$

Proof. It is clear that the difference $\mathcal{A}_\omega(x) - \mathcal{A}_\alpha(x)$ is a purely topological invariant completely determined by the free homotopy class of x in $E \setminus B$. Calculating this difference for the k -times iterated class of the fiber of $M \rightarrow B$ we see that the difference is equal to $-\pi k/2$; see Example 4.2.1. \square

It follows from Lemma 4.2.2 that the Hamiltonian action is bounded from below by $-\pi k/2$ and the question if it can really be negative was raised in, e.g., [44]. Our next proposition gives an affirmative answer to this question.

Proposition 4.2.3. *For every k there exists a contact form $\alpha = f\alpha_0$, where $f > 1/2$, with a closed Reeb orbit x in the class \mathfrak{f}^k such that $\mathcal{A}_\omega(x)$ is arbitrarily close to $-\pi k/2$.*

Proof. Recall that every free homotopy class can be realized by an embedded smooth oriented loop which is tangent to the contact structure; see, e.g., [17] and references therein. Let y be such a loop in the class \mathfrak{f}^k . By moving y slightly in the normal direction to \dot{y} in $\xi = \ker \alpha_0$, i.e., in the direction of $J\dot{x}$ where J is an almost complex structure on ξ compatible with $d\alpha_0$, we obtain a transverse embedded loop x . The loop x is nearly tangent to ξ and we can have

$$0 < \int_x \alpha_0 < \epsilon$$

for an arbitrarily small $\epsilon > 0$.

It is a standard (and easy to prove) fact that there exists a contact form β on M supporting ξ such that x is, up to a parametrization, a closed Reeb orbit of α . Let $g = \alpha_0/\beta$, i.e., g is defined by $\alpha_0 = g\beta$. By scaling β if necessary, we can ensure that $g \leq 1$. In other words, $1/g \geq 1$. It is not hard to see that $g|_x$ extends to a function h on M so that $h/g > 1/2$ and the derivative of h in the normal direction to x is zero, i.e., $\ker dh \supset \xi$ at all points of x .

The latter condition guarantees that x is still a closed Reeb orbit of $\alpha := h\beta = f\alpha_0$, where $f = h/g$. By construction, $f > 1/2$. Furthermore, we have $\alpha|_x = \alpha_0|_x$, and hence

$$0 < \int_x \alpha = \int_x \alpha_0 < \epsilon.$$

Thus, by (4.2.1), $-\pi k/2 < \mathcal{A}_\omega(x) \leq \epsilon - \pi k/2$ and $\mathcal{A}_\omega(x)$ can be made arbitrarily close to $-\pi k/2$. \square

4.2.1 Applications of homology vanishing to prequantization bundles

When B is aspherical, $\text{SH}(W) = 0$ for $W = W_f$ by the Künneth formula from [44] and the results proved in Section 4.1 directly apply to W .

For instance, combining the Thom isomorphism $\text{H}(B) = \text{H}(W, \partial W)[-2]$ with Proposition 4.1.1, we obtain

Corollary 4.2.4. *Assume that B^{2m} is symplectically aspherical and $W = W_f$. Then we have natural isomorphisms*

$$(i) \text{SH}^-(W) = \text{H}(B)[-m + 1] \text{ and } \text{SH}^{-,S^1}(W) = \text{H}(B) \otimes \text{H}(\mathbb{C}\mathbb{P}^\infty)[-m + 1];$$

$$(ii) \text{SH}^+(W) = \text{H}(B)[-m + 2] \text{ and}$$

$$\text{SH}^{+,S^1}(W) = \text{H}(B) \otimes \text{H}(\mathbb{C}\mathbb{P}^\infty)[-m + 2]; \quad (4.2.2)$$

(iii) *and, under the identification (4.2.2), the Gysin sequence shift map*

$$\text{SH}_{r+2}^{+,S^1}(W) \xrightarrow{D} \text{SH}_r^{+,S^1}(W)$$

is the identity on the first factor and the map $\text{H}_{q+2}(\mathbb{C}\mathbb{P}^\infty) \rightarrow \text{H}_q(\mathbb{C}\mathbb{P}^\infty)$, given

by the pairing with a suitable chosen generator of $H^2(\mathbb{C}\mathbb{P}^\infty)$, on the second. In particular, D is an isomorphism when $r \geq m + 2$.

Remark 4.2.5. Recall that even when B is not aspherical but simply meets the standard conditions, e.g., that E is weakly monotone, sufficient to have the (equivariant) symplectic homology of W_f defined, the homology is independent of f ; see [61] and also [50].

Likewise, and Lemma 4.1.2 and Proposition 4.1.6 yeild

Corollary 4.2.6. *Assume that B is symplectically aspherical. Then, for every $\zeta \in H_d(B)$ and $W = W_f$, we have*

$$0 \leq c_{\zeta,0}^{S^1}(W) \leq c_{\zeta,1}^{S^1}(W) \leq c_{\zeta,2}^{S^1}(W) \leq \dots \text{ and } c_{\zeta,0}^{S^1}(W) \leq c_\zeta(W), \quad (4.2.3)$$

and, when $2k \geq 2m - d$,

$$0 \leq c_{\zeta,k+1}^{S^1}(W) - c_{\zeta,k}^{S^1}(W) \leq c(W).$$

Moreover, $c_{\zeta,0}^{S^1}(W) > 0$, and hence all capacities are strictly positive.

The new point here, when compared to the general results, is the last assertion that the capacities are strictly positive. To see this, note first that these capacities are monotone (with respect to inclusion) on the domains W_f . Thus it suffices to show that $c_{\zeta,0}^{S^1}(U) > 0$ for a small tubular neighborhood U of B in E bounded by the S^1 -bundle $r = \epsilon$. It is not hard to see that in this case $c_{\zeta,0}^{S^1}(U) = \mathcal{A}_\omega(x)$ for a closed Reeb orbit x on M ; see Section 5.2. Hence, by Example 4.2.1, $c_{\zeta,0}^{S^1}(U) \geq \pi\epsilon^2/2 > 0$.

4.2.2 Stable displacement

The zero section of a prequantization bundle E is never topologically displaceable since its intersection product with itself is Poincaré dual, up to a non-zero factor, to $[\sigma] \neq 0$. As a consequence, no compact subset containing the zero section is topologically displaceable either. However, the situation changes dramatically when one considers stable displaceability.

Proposition 4.2.7. *The zero section is stably displaceable in E .*

Proof. The zero section B is a symplectic submanifold of E . Thus $M := B \times S^1$ is nowhere coisotropic in $E \times T^*S^1$, i.e., at no point the tangent space to M is coisotropic. Furthermore, M is smoothly infinitesimally displaceable: there exists a non-vanishing vector field along M which is nowhere tangent to M . Now the proposition follows from [30, Theorem 1.1]. (When $\dim B = 2$ one can also use the results from [39, 45]). \square

Remark 4.2.8. This argument shows that every closed symplectic submanifold M of any symplectic manifold is stably displaceable.

As a consequence of Proposition 4.2.7, a sufficiently small tubular neighborhood of M is also stably displaceable in E . However, in contrast with the case of Liouville manifolds, this is not enough to conclude that arbitrary large tubular neighborhoods, and thus all compact subsets of E , are also stably displaceable and in fact they need not be.

Example 4.2.9. Let E be the tautological bundle over $\mathbb{C}\mathbb{P}^1$, i.e., a blow up of \mathbb{C}^2 . Then E contains a monotone torus L which is the restriction of the S^1 -bundle (for a suitable radius) to the equator. It is known that $\text{HF}(L, L) \neq 0$; [57, Section 4.4]. Then, by the Künneth formula, $\text{HF}(L', L') \neq 0$ where $L' = L \times S^1$ in $E \times T^*S^1$. See

[51, 59] for generalizations of this example and its connections with non-vanishing of symplectic homology.

Remark 4.2.10. Note that Proposition 4.2.7 holds for any base B , but gives no information about $\text{SH}(W)$. The reason is that the Künneth formula does not directly apply in this case even when B is aspherical; see [43]. The boundary of a tubular neighborhood U of $B \times S^1$ in $E \times T^*S^1$ does not have contact type and, in fact, the symplectic form is not even exact near the boundary. As a consequence, the symplectic homology of U is not defined. One can still introduce an *ad hoc* variant of such a homology to have the Künneth formula and then reason along the lines of the proof of Theorem 4.1.9 to show that this homology, and hence $\text{SH}(W)$, vanishes. However, the argument is not much simpler than the proof in [44]. It is also worth pointing out that $\text{SH}(W)$ might or might not vanish depending on B ; [50, 51, 59].

On the other hand, the proposition does imply, via the Künneth formula, vanishing of the Rabinowitz Floer homology for low energy levels in E , i.e., for r close to zero, proved originally in [2]. Note in this connection that, as was pointed out to us by Alex Oancea, the Rabinowitz Floer homology might depend on the level in this case.

Proposition 4.2.7 has some standard consequences along the lines of the almost existence theorem, the Weinstein conjecture and lower bounds on the growth of periodic orbits, which all are proved via variants of the displacement energy–capacity inequalities and are accessible by several methods requiring somewhat different assumptions on (B, σ) ; see, e.g., [46]. Here, dealing with the almost existence, we adopt the setting from [55] which is immediately applicable.

The condition which E must satisfy then is that it is *stably strongly semi-positive* in the sense of [55], which is the case if and only if $N_B \geq n + 1$ or (B^{2n}, σ)

is positive monotone, i.e., $c_1(TB) = \lambda[\omega]$ on $\pi_2(B)$ where $\lambda \geq 0$ and in addition we require that $\langle [\omega], \pi_2(B) \rangle = 0$ whenever $\langle c_1(TB), \pi_2(B) \rangle = 0$. (Here N_B is the minimal Chern number of B . Note that $N_E = N_B - 1$.)

Corollary 4.2.11 (Almost Existence in E near B ; [40]). *Assume that B is as above. Let H be a smooth, proper, autonomous Hamiltonian on E and let I be a (possibly empty) interval such that $\{H = c\}$ is contained in a sufficiently small neighborhood of B in E . Then, for almost all $c \in I$ in the sense of measure theory, the level $\{H = c\}$ carries a periodic orbit of H .*

This is an immediate consequence of the displacement energy–capacity inequality from [55] and [41, Theorem 2] which allows us to avoid imposing the extra condition that H is bound from below or above. The corollary is not the most general result of this kind. It is a particular case of the main theorem from [40]. However, our proof is simpler than the argument *ibid* and, in fact, Remark 4.2.8 can be used to simplify some parts of that argument. Corollary 4.2.11 implies the Weinstein conjecture for contact type hypersurfaces in E near B . There are, of course, many other instances where the Weinstein conjecture is known to hold for hypersurfaces in E . For example, although to the best of knowledge it is still unknown if it holds in general for prequantization bundles, it does hold under suitable additional conditions. For instance, this is the case when σ is aspherical, [44], or more generally if $\pi^*[\sigma]$ is nilpotent in the quantum cohomology of E , [50].

The second application is along the lines of the Conley conjecture (see [24]) or [60, Prop. 4.13] and concerns the number or the growth of simple periodic orbits of compactly supported Hamiltonian diffeomorphisms. For the sake of simplicity we assume that σ is aspherical although this condition can be significantly relaxed.

Let $H: S^1 \times E \rightarrow \mathbb{R}$ be a compactly supported Hamiltonian.

Corollary 4.2.12. *Assume that σ is symplectically aspherical and $\text{supp } H$ is con-*

tained in a sufficiently small neighborhood of B in E . Then φ_H has infinitely many simple contractible periodic orbits with non-zero action provided that $\varphi_H \neq id$. Moreover, when $H \geq 0$ the number of such orbits of period up to k and with positive action grows at least linearly with k unless, of course, $H = 0$.

Here the first assertion follows readily from [19, Theorem 2] and the second assertion is a consequence of [30, Theorem 1.2].

Chapter 5

Linking number filtration

In this section we construct and utilize a certain additional filtration on the positive (equivariant) symplectic homology of a prequantization disk bundle $W \rightarrow B$. This filtration is, roughly speaking, given by the linking number of a closed Reeb orbit and the zero section. It “commutes” with the Hamiltonian action filtration and plays essentially the same role as the grading by the free homotopy class of the fiber in the contact homology of the corresponding circle bundle. Although the linking number filtration can be defined in a more general setting, it is of particular interest to us when the base B is symplectically aspherical. This is the assumption we will make henceforth. We will then use the linking number filtration to reprove the non-degenerate case of the contact Conley conjecture, originally established in [26, 27], without relying on the machinery of contact homology.

5.1 Definition of the linking number filtration

Throughout this section we keep the notation and convention from Sections 2.3 and 4.2. In particular, E and M are the prequantization line and circle

bundles, respectively, over a symplectically aspherical manifold (B^{2m}, σ) . Let f be a function on E such that $f > 1/2$. The domain W_f is bounded by the fiberwise star-shaped hypersurface given by $(1 + r^2)/2 = f$. This hypersurface M_f is of contact type and the restriction of the primitive $(1 + r^2)\alpha_0/2$ to M_f is $\alpha = f\alpha_0$. Hence, we will also use the notations M_α and W_α for the hypersurface M_f and the domain W_f . Furthermore, recall that all loops and periodic orbits we consider are assumed to be contractible in E unless stated otherwise.

Assume that α is nondegenerate. Let H is a time-dependent admissible Hamiltonian on $E = \widehat{W}_f$. We require H to be constant on a neighborhood U of B . As well, it is required that all one-periodic orbits x of H outside U are small perturbations of closed Reeb orbits. We call these orbits non-constant. For a generic choice of such a Hamiltonian H , all non-constant orbits are nondegenerate.

Fix an almost complex structure J on E compatible with ω , which is independent of time near B and outside a large compact set, and such that B is an almost complex submanifold of E . Consider solutions $u: \mathbb{R} \times S^1 \rightarrow E$ of the Floer equation for (H, J) asymptotic as $s \rightarrow \pm\infty$ to non-constant orbits. By the results from [18] the regularity conditions are satisfied for a generic pair (H, J) meeting the above requirements. With this in mind, we have the complex $\text{CF}^+(H)$ generated by non-constant one-periodic orbits of H , contractible in E , and equipped with the standard Floer differential. Clearly, the homology of this complex is $\text{HF}^+(H)$.

The complex $\text{CF}^+(H)$ carries a natural filtration by the linking number with B . Indeed, since H is constant near B , every solution u of the Floer equation is a holomorphic curve near B . Then, by the assumption that B is an almost complex submanifold of E , the intersection index of u with B is non-negative. When u is a solution connecting x to y , the difference $L_B(x) - L_B(y)$ is exactly this intersection

number. Thus

$$L_B(x) \geq L_B(y) \tag{5.1.1}$$

and the Floer differential does not increase L_B . In other words, for every $k \in \mathbb{Z}$, the subspace $\text{CF}^+(H, \mathfrak{f}^{\leq k})$ generated by the orbits x with $L_B(x) \leq k$ is a subcomplex and we obtain an increasing filtration of the complex $\text{CF}^+(H)$. Set

$$\text{CF}^+(H, \mathfrak{f}^k) := \text{CF}^+(H, \mathfrak{f}^{\leq k}) / \text{CF}^+(H, \mathfrak{f}^{\leq k-1}).$$

We denote the homology of the resulting complexes by $\text{HF}^+(H, \mathfrak{f}^{\leq k})$ and, respectively, $\text{HF}^+(H, \mathfrak{f}^k)$.

Passing to the direct limit over H , we obtain the homology groups $\text{SH}^+(W, \mathfrak{f}^{\leq k})$ and, $\text{SH}^+(W, \mathfrak{f}^k)$, which fit into a long exact sequence

$$\dots \rightarrow \text{SH}^+(W, \mathfrak{f}^{\leq k-1}) \rightarrow \text{SH}^+(W, \mathfrak{f}^{\leq k}) \rightarrow \text{SH}^+(W, \mathfrak{f}^k) \rightarrow \dots$$

Furthermore, the complexes $\text{CF}^+(H, \mathfrak{f}^{\leq k})$ and $\text{CF}^+(H, \mathfrak{f}^k)$ inherit the filtration by the Hamiltonian action from the complex $\text{CF}(H)$ and this filtration descends to the resulting homology groups.

The construction extends to the equivariant setting in a straightforward way. Let $\tilde{H}: S^1 \times E \times S^{2m+1} \rightarrow \mathbb{R}$ be a parametrized Hamiltonian. Assume that \tilde{H} is a small perturbation of an ordinary non-degenerate Hamiltonian $H: S^1 \times E \rightarrow \mathbb{R}$. Then x in such a pair (x, z) is small perturbation of a one-periodic orbit of H for a critical point (x, z) of the parametrized action functional $\mathcal{A}_{\tilde{H}}$. As defined in Remark 3.2.6, we have the quotient complex $\text{CF}^{+,S^1,m}(\tilde{H})$. The essential point is that again the Hamiltonian \tilde{H} can be taken constant on U on every slice $E \times (t, z)$, where t and z are points on S^1 and S^{2m+1} , respectively. Then the quotient complex $\text{CF}^{+,S^1,m}(\tilde{H})$ is generated by the critical orbit $S_{(x,z)}$ such that x is a non-constant

orbit.

Recall that the parametrized Floer equation has the form

$$\begin{aligned} \partial_s u + J \partial_t u &= \nabla_E \tilde{H}, \\ \frac{d\lambda}{ds} &= \int_{S^1} \nabla_z \tilde{H}(u(t, s), t, \lambda(s)) dt, \end{aligned}$$

where $\lambda: \mathbb{R} \rightarrow S^{2m+1}$ and $u: S^1 \times \mathbb{R} \rightarrow E$. Thus, when \tilde{H} is constant and the projection u to E of a solution (u, λ) is a holomorphic curve near B . It follows that (5.1.1) holds when (u, λ) connects a critical family containing x to a critical family containing y . As a consequence, the complex $\text{CF}^{+, S^1, m}(\tilde{H})$ is filtered by the linking number. Passing to the limit as $m \rightarrow \infty$ and then over \tilde{H} we obtain the linking number filtration on $\text{SH}^{+, S^1}(W)$.

We denote the resulting equivariant symplectic homology groups by $\text{SH}^{+, S^1}(W, \mathfrak{f}^{\leq k})$ and $\text{SH}^{+, S^1}(W, \mathfrak{f}^k)$. As in the non-equivariant case, these groups fit into the long exact sequence

$$\dots \rightarrow \text{SH}^{+, S^1}(W, \mathfrak{f}^{\leq k-1}) \rightarrow \text{SH}^{+, S^1}(W, \mathfrak{f}^{\leq k}) \rightarrow \text{SH}^{+, S^1}(W, \mathfrak{f}^k) \rightarrow \dots$$

and also inherit the action filtration. Moreover, it is easy to see that the shift operator D from the Gysin sequence respects the linking number filtration.

It is clear that the linking number filtration is preserved by the continuation maps because the continuation Hamiltonians can also be taken constant on U . As a consequence, the resulting groups are independent of the original contact form α or, equivalently, the domain W_α .

5.2 Calculation of the homology groups

The calculation of the linking number filtration groups $\mathrm{SH}^+(W, \mathfrak{f}^k)$ and $\mathrm{SH}^{+,S^1}(W, \mathfrak{f}^k)$ is based on the standard Morse–Bott type argument in Floer homology.

Proposition 5.2.1. *Let, as above, W be the prequantization disk bundle over a symplectically aspherical manifold (B^{2m}, σ) . Then*

- (i) $\mathrm{SH}^+(W, \mathfrak{f}^k) = 0$ and $\mathrm{SH}^{+,S^1}(W, \mathfrak{f}^k) = 0$ for $k \leq 0$,
- (ii) $\mathrm{SH}^+(W, \mathfrak{f}^k) = \mathrm{H}_*(M)[2k - m]$ and
- (iii) $\mathrm{SH}^{+,S^1}(W, \mathfrak{f}^k) = \mathrm{H}_*(B)[2k - m]$ for $k \in \mathbb{N}$,

where all homology groups are taken with rational coefficients.

In particular,

$$\mathrm{SH}_{2k+m}^{+,S^1}(W, \mathfrak{f}^k) = \mathbb{Q} \text{ for } k \in \mathbb{N} \quad (5.2.1)$$

and, as expected, $\mathrm{SH}^{+,S^1}(W, \mathfrak{f}^k)$ is isomorphic to the contact homology groups of (M, ξ) for the free homotopy class \mathfrak{f}^k , cf. [10].

Proof. Consider an admissible Hamiltonian of the form $H = h(r^2)$, where h is monotone increasing, convex function equal to zero on $[0, 1 - \epsilon]$ and to $ar^2 + b$ on $[1 + \epsilon, \infty)$. The non-trivial one-periodic orbits of H occur on Morse–Bott non-degenerate levels $r = r_1, \dots, r_l$, where $l = \lfloor a/\pi \rfloor$, when the form α_0 is normalized to have integral π over the fiber. The linking number of the orbits on the level $r = r_k$ with B is exactly k . Now the proposition follows by the standard Morse–Bott argument in Floer homology (see, e.g., [5, 48] and also [25]) together with an index calculation as in, e.g., [22]. \square

Comparing Case (iii) of Proposition 5.2.1 and Case (ii) of Proposition 4.2.4, we see that

$$\mathrm{SH}^{+,S^1}(W) = \bigoplus_{k \in \mathbb{N}} \mathrm{SH}^{+,S^1}(W, \mathfrak{f}^k) \quad (5.2.2)$$

although the isomorphism is not canonical in contrast with (4.2.2). Note that there is no similar isomorphism in the non-equivariant case: $\mathrm{SH}^+(W) \neq \bigoplus_{k \in \mathbb{N}} \mathrm{SH}^+(W, \mathfrak{f}^k)$ and, in fact, the sum on the right is much bigger than $\mathrm{SH}^+(W)$.

Remark 5.2.2. Although this is not immediately obvious, one can expect the natural maps $\mathrm{SH}^{+,S^1}(W, \mathfrak{f}^{\leq k}) \rightarrow \mathrm{SH}^{+,S^1}(W)$ to be monomorphisms, resulting in a filtration of $\mathrm{SH}^{+,S^1}(W)$ by the groups $\mathrm{SH}^{+,S^1}(W, \mathfrak{f}^{\leq k})$. Then the right hand side of (5.2.2) would be the graded space associated with this filtration. On the other hand, the decomposition (4.2.2) gives rise to the filtration $\bigoplus_{q \leq k} \mathrm{H}_*(B) \otimes \mathrm{H}_{2q}(\mathbb{C}\mathbb{P}^\infty)$, and the two filtrations appear to be different. Namely, the shift operator D is strictly decreasing with respect to the filtration coming from (4.2.2) and thus the induced operator on the graded space is zero. However, under the identification $\mathrm{SH}^+(W, \mathfrak{f}^k) \cong \mathrm{H}_*(B)$ from Case (ii) of Proposition 5.2.1 the operator D is given by pairing with $[\sigma] \in \mathrm{H}^2(B)$ (see [25, Proposition 2.22]) and this pairing is non-trivial. To put this somewhat informally, the decompositions (4.2.2) and (5.2.2) do not match term-wise.

Remark 5.2.3 (Lusternik–Schnirelmann inequalities). We can also use the linking number filtration to extend the Lusternik–Schnirelmann inequalities from [25, Theorem 3.4] to prequantization bundles. Namely, assume that all closed Reeb orbits on M_α are isolated. Then, for any $\beta \in \mathrm{SH}^{+,S^1}(W)$, we have

$$c(\beta, W_\alpha) > c(D(\beta), W_\alpha), \quad (5.2.3)$$

where the right hand side is by definition $-\infty$ when $D(\beta) = 0$. In particular,

when the orbits are isolated,

$$0 < c_{\zeta,0}^{S^1}(W) < c_{\zeta,1}^{S^1}(W) < c_{\zeta,2}^{S^1}(W) < \dots$$

in (4.2.3) for every $\zeta \in H_*(B)$. For the sake of brevity we only outline the proof of (5.2.3). Consider a “sufficiently large” admissible autonomous Hamiltonian H constant on U . Then, by Theorem 3.3.2, the strict Lusternik–Schnirelmann inequality holds for H . As a consequence, there exist two one-periodic orbits x and y of H contractible in E , the carriers for the corresponding action selectors for β and $D(\beta)$ in $\text{HF}^{+,S^1}(H)$, such that $\mathcal{A}_H(x) > \mathcal{A}_H(y)$ and x and y are connected by a solution u of the Floer equation. As in the proof of [25, Theorem 3.4], we need to show that this inequality remains strict as we pass to the limit. When x and y are in the same free homotopy class (i.e., $L_B(x) = L_B(y)$), that proof goes through word-for-word. When, $L_B(x) > L_B(y)$, the Floer trajectory u has to cross U , where it is a holomorphic curve, passing through a point of B . By the standard monotonicity argument, $\mathcal{A}_H(x) - \mathcal{A}_H(y) = E(u) > \epsilon > 0$, where $E(u)$ is the energy of u and ϵ is independent of H .

Chapter 6

Contact Conley conjecture

6.1 Local symplectic homology

In this section we recall the definitions of the (equivariant) local symplectic homology and of symplectically degenerate maxima (SDM) for Reeb flows – the ingredients essential for the statement and the proof of the non-degenerate case of the contact Conley conjecture.

Let x be an isolated closed Reeb orbit of period T , not necessarily simple, for a contact form α on M^{2m+1} . The Reeb vector field coincides the Hamiltonian vector field of the Hamiltonian r on $M \times (1 - \epsilon, 1 + \epsilon)$ equipped with the symplectic form $d(r\alpha)$. Consider now the Hamiltonian $H = T \cdot h(r)$, where $h'(1) = 1$ and $h''(1) > 0$ is small. On the level $r = 1$, this flow is simply a reparametrization of the Reeb flow and the orbit x corresponds to an isolated one-periodic orbit \tilde{x} of H . By definition, the *equivariant local symplectic homology* $\text{SH}^{S^1}(x)$ of x is the local S^1 -equivariant Floer homology $\text{HF}^{S^1}(\tilde{x})$ of \tilde{x} ; see [25, Section 2.3]. It is easy to see that $\text{SH}^{S^1}(x) := \text{HF}^{S^1}(\tilde{x})$ is independent of the choice of the function h . Note also that this construction is purely local: it only depends on the germ of α along x . In what follows, we will use the notation x for both orbits \tilde{x} and x .

These local homology groups do not carry an absolute grading by the Conley–Zehnder index. To fix such a grading, it is enough to pick a symplectic trivialization of $T(M \times (1 - \epsilon, 1 + \epsilon))|_x$. Depending on a specific setting, there can be different natural ways to do this. For instance, one can start with a trivialization of the contact structure $\xi = \ker \alpha$ along x ; for this trivialization naturally extends to a trivialization of $T(M \times (1 - \epsilon, 1 + \epsilon))|_x$. However, we are interested in the setting where M has an aspherical filling W and x is contractible in W . Then it is more convenient to obtain a trivialization of

$$T(M \times (1 - \epsilon, 1 + \epsilon))|_x = TW|_x$$

from a capping of x in W . In any event, when x is iterated, i.e., $x = y^k$ where y is simple and also contractible in W , we will always assume that the trivialization of $TW|_x$ comes from a trivialization along y . This is essential to guarantee that the mean index is homogeneous under iterations: $\hat{\mu}(x) = k\hat{\mu}(y)$.

Example 6.1.1. Assume that x is non-degenerate. Then $\mathrm{SH}^{S^1}(x) = \mathbb{Q}$, concentrated in degree $\mu(x)$, when x is good; and $\mathrm{SH}^{S^1}(x) = 0$ when x is bad; see [25, Section 2.3] and, in particular, Examples 2.18 and 2.19 therein.

Furthermore, it is worth keeping in mind that \tilde{x} is degenerate even when x is non-degenerate. Indeed, the linearized flow along \tilde{x} has 1 as an eigenvalue and its algebraic multiplicity is at least 2.

As a consequence, $\mathrm{SH}^{S^1}(x)$ is supported in the interval of length $2m$ centered at the mean index $\hat{\mu}(x)$ of x , i.e., only for the degrees in this range the homology can be non-zero. (If \tilde{x} were non-degenerate the length of the interval would be $2m + 2$.) In other words, using self-explanatory notation, we have

$$\mathrm{supp} \mathrm{SH}^{S^1}(x) \subset [\hat{\mu}(x) - m, \hat{\mu}(x) + m]; \quad (6.1.1)$$

see [25, Proposition 2.20]. Moreover,

$$\text{supp SH}^{S^1}(x) \subset (\hat{\mu}(x) - m, \hat{\mu}(x) + m) \quad (6.1.2)$$

when x is *weakly non-degenerate*, i.e., at least one of its Floquet multipliers is different from 1.

Remark 6.1.2. Conjecturally, when as above x is the k th iteration of a simple orbit,

$$\text{SH}^{S^1}(x) \cong \text{HC}(x) \cong \text{HF}(\varphi)^{\mathbb{Z}_k}, \quad (6.1.3)$$

where $\text{HC}(x)$ is the local contact homology of x introduced in [35] (see also [28]), φ is the return map of x , and $\text{HF}(\varphi)^{\mathbb{Z}_k}$ is the \mathbb{Z}_k -invariant part in the local Floer homology of φ with respect to the natural \mathbb{Z}_k -action. When x is simple, i.e., $k = 1$, this has been proved. Indeed, in this case, $\text{HC}(x)$ is rigorously defined and the first isomorphism is a local version of the main result in [10]. The second isomorphism is established in [35] and can also be thought of as a local variant of the isomorphism between Floer and contact homology in [16]. When $k \geq 1$, there are foundational problems with the construction of $\text{HC}(x)$ common to many versions of the contact homology (see, however, [42]) and proving directly that the first and the last term in (6.1.3) are isomorphic might be a simpler approach. We will return to this question elsewhere.

When M is compact, the groups $\text{SH}^{S^1}(x)$, where x ranges over all closed Reeb orbits of α (not necessarily simple), are the building blocks for $\text{SH}^{+,S^1}(W)$ where W is a symplectically aspherical filling of M . (For instance, vanishing of the local homology groups for all x implies vanishing of the global homology.) However, there might be a shift of degrees which depends on the choice of trivializations along the orbits x . This shift is obviously zero when x is contractible in

W and the trivialization of $T_x W$ comes from a capping of x . The same holds for filtered (by action or linking number) homology groups. In particular, we have

Lemma 6.1.3. *Let α be a contact form on the prequantization circle bundle over a symplectically aspherical manifold (B^{2m}, σ) . Assume that all closed Reeb orbits x in the class \mathfrak{f}^k are isolated and $\mathrm{SH}_q^{S^1}(x) = 0$ for all such x with respect to the trivialization of $T_x W$ coming from a capping of x in W . Then $\mathrm{SH}_q^{+,S^1}(W, \mathfrak{f}^k) = 0$.*

The lemma readily follows from the observation that under the above conditions $\mathrm{SH}_q^{+,S^1}(H, \mathfrak{f}^k) = 0$ for a suitable cofinal family of admissible Hamiltonians H . (More generally, there is a spectral sequence starting with $\bigoplus_x \mathrm{SH}^{S^1}(x)$ and converging to $\mathrm{SH}^{+,S^1}(W)$, which also implies the lemma. We do not need this fact and we omit its proof for the sake of brevity, for it is quite standard; see, e.g., [26] where such a spectral sequence is constructed for the contact homology.)

Next recall that an iteration k of x is called *admissible* when none of the Floquet multipliers of x , different from 1, is a root of unity of degree k . For instance, every k is admissible when no Floquet multiplier is a root of unity or, as the opposite extreme, when x is totally degenerate, i.e., all Floquet multipliers are equal to 1. Furthermore, every sufficiently large prime k (depending on x) is admissible.

For our purposes it is convenient to adopt the following definition. Namely, x is a *symplectically degenerate maximum (SDM)* if there exists a sequence of admissible iterations $k_i \rightarrow \infty$ such that

$$\mathrm{SH}_q^{S^1}(x^{k_i}) \neq 0 \text{ for } q = \hat{\mu}(x^{k_i}) + m = k_i \hat{\mu}(x) + m. \quad (6.1.4)$$

This condition is obviously independent of the choice of a trivialization along x . It follows from (6.1.2) that then x^{k_i} , and hence x , must be totally degenerate.

Thus the definition can be rephrased as that x is totally degenerate and (6.1.4) holds for some sequence $k_i \rightarrow \infty$.

Remark 6.1.4. Continuing the discussion in Remark 6.1.2 note there are several, hypothetically equivalent, ways to define a closed SDM Reeb orbit. The definition above is a contact analog of the original definition of a Hamiltonian SDM from [20] and it lends itself conveniently to the proof of the non-degenerate case of the contact Conley conjecture. Alternatively, a contact SDM was defined in [28] as a closed isolated Reeb orbit x with $\text{HC}_{\hat{\mu}(x)+m}(x) \neq 0$. By (6.1.3), its contact analogue would be that $\text{SH}_{\hat{\mu}(x)+m}^{S^1}(x) \neq 0$. For a simple orbit this is equivalent to that the fixed point of φ is an SDM. Furthermore, one can show that the \mathbb{Z}_k -action on the homology is trivial for totally degenerate orbits and thus $\text{HF}(\varphi)^{\mathbb{Z}_k} = \text{HF}(\varphi)$. Hence, the equivalence of the two definitions would then follow from the identification of the first and the last term in (6.1.3) combined with the persistence of the local Floer homology, [23].

6.2 Conley conjecture

Now we are in the position to state and prove the non-degenerate case of the contact Conley conjecture, a contact analogue of the main result from [53].

Theorem 6.2.1 (Contact Conley Conjecture). *Let $M \rightarrow B$ be a prequantization S^1 -bundle and let α be a contact form on M supporting the standard (co-oriented) contact structure ξ on M . Assume that*

- (i) B is symplectically aspherical,
- (ii) $\pi_1(B)$ is torsion free.

Then the Reeb flow of α has infinitely many simple closed Reeb orbits with contractible projections to B , provided that none of the orbits in the free homotopy

class \mathfrak{f} of the fiber is an SDM. Assume in addition that the Reeb flow of α has finitely many closed Reeb orbits in the class \mathfrak{f} . Then for every sufficiently large prime k the Reeb flow of α has a simple closed orbit in the class \mathfrak{f}^k .

Before proving this theorem let us compare it with other results on the contact Conley conjecture. Theorem 6.2.1 was proved in [26, 27] without the assumption that none of the orbits is an SDM. However, that argument relied on the machinery of linearized contact homology which is yet to be put on a completely rigorous foundation. (See, however, [42] where some of the foundational issues have been resolved in dimension three.) The key difficulty in translating the proof from those two papers into the symplectic homology framework in the non-degenerate case was purely conceptual: the grading by the free homotopy classes \mathfrak{f}^k is crucial for the proof and while the cylindrical contact homology is graded by \mathfrak{f}^k the symplectic homology is not. The linking number filtration is an analogue of this grading in symplectic homology which allows us to overcome this problem. On the other hand, removing the “non-SDM” assumption requires replacing the contact homology by the symplectic homology in the main result of [28]. This is a non-trivial but technical issue and we will return to it elsewhere.

There are also some minor discrepancies between the assumptions of Theorem 6.2.1 and its counterpart in [26, 27]. Namely, there the base B is assumed to be aspherical, i.e., $\pi_r(B) = 0$ for $r \geq 2$, but as is pointed out in [26, Section 2.2], this assumption is only used to make sure that σ is aspherical and $\pi_1(B)$ is torsion free. Then, it is also required there that the class $c_1(\xi)$ be atoroidal. This is a minor technical restriction imposed only for the sake of simplicity and it does not arise in the symplectic homology setting because the fiber is contractible in W .

Proof of Theorem 6.2.1. The argument closely follows the reasoning in [26] which

in turn is based on the proof in [53]. We need the following simple and purely algebraic fact, proved in [26, Lemma 4.2], which only uses the conditions that ω is aspherical and that $\pi_1(B)$ is torsion free.

Lemma 6.2.2. *Under the conditions of the theorem, for every $k \in \mathbb{N}$ the only solutions $\mathfrak{h} \in \tilde{\pi}_1(P)$ and $l \geq 0$ of the equation $\mathfrak{h}^l = \mathfrak{f}^k$ are $\mathfrak{h} = \mathfrak{f}^r$, for some $r \in \mathbb{N}$, and $l = k/r$. (In particular, \mathfrak{f} is primitive.)*

Next, without loss of generality we may assume that there are only finitely many closed Reeb orbits in the class \mathfrak{f} , for otherwise there is nothing to prove. We denote these orbits by x_1, \dots, x_r and set $\Delta_j = \hat{\mu}(x_j)$, where we equipped $T_{x_i}W$ with a trivialization coming from a capping of x_i in W . Let k be a large prime. Then, unless there is a simple closed Reeb orbit in the class \mathfrak{f}^k , every closed Reeb orbit in this class has the form x_j^k by Lemma 6.2.2.

We will show that in this case $\text{SH}_{m+2k}^{+,S^1}(W, \mathfrak{f}^k) = 0$, when k is large, which contradicts Proposition 5.2.1 and more specifically (5.2.1). By Lemma 6.1.3, it is enough to prove that

$$\text{SH}_{m+2k}^{S^1}(x_j^k) = 0. \quad (6.2.1)$$

Pick the prime k so large that $k|\Delta_j - 2| > 2m$ for all x_j with $\Delta_j \neq 2$. Then, since $\hat{\mu}(x_j^k) = k\Delta_j$, we have

$$\text{supp SH}^{S^1}(x_j^k) \subset [k\Delta_j - m, k\Delta_j + m]$$

by (6.1.1), and hence $m + 2k$ is not in the support. Thus (6.2.1) holds in this case. On the other hand, when $\Delta_j = 2$, (6.2.1) holds when k is sufficiently large, for otherwise (6.1.4) would be satisfied for some sequence of primes $k_i \rightarrow \infty$ and x_j would be an SDM. This completes the proof of the theorem. \square

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