THE LEGENDRIAN SEIFERT CONJECTURE: ARNOLD'S PROBLEM 1994-13

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Problem 1994-13: *Consider a particle in a magnetic field on a surface M^2. Study Legendrian divergence–free vector fields on ST^*M^2 and, in particular, their closed orbits. More generally, consider divergence–free Legendrian vector fields on S^3 for some (e.g., standard) contact structure. Does there exist a counterexample to the Seifert conjecture in this class of vector fields?* See: V.I. Arnol'd, *Zadachi Arnol'da* (Arnold problems), preface by M.B. Sevryuk and V.B. Filippov, Izdatel'stvo FAZIS, Moscow, 2000 (in Russian). The comments below are prepared for the English translation of this book.

Comments: The Seifert conjecture is the question posed by Seifert, [Se], whether or not every smooth non-vanishing vector field on the 3-dimensional sphere has a periodic orbit. Of course, a similar question can be asked for other manifolds or some special classes of vector fields (e.g., of a certain smoothness class, volume–preserving, Hamiltonian, Legendrian, etc.). Traditionally, non-trivial examples of non-vanishing vector fields without periodic orbits are referred to as *counterexamples* to the Seifert conjecture.

A C^1-smooth counterexample to the Seifert conjecture on S^3 was constructed by Schweitzer, [Sc]. The C^1-smoothness constraint was later improved to $C^{2+\alpha}$ by Harrison, [Ha]. Finally, a C^∞-counterexample was found by K. Kuperberg, [KuK1]; see also [KuGK]. A construction of a C^1-smooth volume–preserving counterexample on S^3 is carried out by G. Kuperberg, [KuG]. To date, it is not known whether or not there exists such a C^∞-smooth counterexample.

The vector field arising in the magnetic problem is not only Legendrian and volume–preserving but also Hamiltonian. The Hamiltonian Seifert conjecture is the question whether or not there exists a proper function on $\mathbb{R}^{2n}$ whose Hamiltonian flow has no periodic orbits on at least one regular level set; such a C^2-smooth function on $\mathbb{R}^4$ is constructed in [GG1, GG2]. As a consequence, it is not hard to see that in the twisted cotangent bundle of S^2 there exists a compact C^2-hypersurface enclosing the zero section and having no closed characteristics.

Counterexamples to the Seifert conjecture are much easier to find in higher dimensions. (Hence, this is perhaps where one should start the investigation of the Legendrian Seifert conjecture.) For $2n+1 \geq 5$, a C^∞-smooth counterexample to the Seifert conjecture on S^{2n+1} was found by Wilson, [Wi]; volume–preserving and Hamiltonian C^∞-smooth counterexamples have also been constructed (see [Gi1, Gi2, He, Ke]). We refer the reader to the surveys [Gi3, Gi4, KuK2, KuK3] for further references and details.

It is not known whether any of these counterexamples can be made Legendrian.

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