A CHARGE IN A MAGNETIC FIELD: ARNOLD'S PROBLEMS 1981-9, 1982-24, 1984-4, 1994-14, 1994-35, 1996-17, AND 1996-18

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1. Problems

The following problems are from *Zadachi Arnol'da* (Arnol'd's problems) by V.I. Arnol'd, preface by M.B. Sevryuk and V.B. Filippov, Izdatel'stvo FAZIS, Moscow, 2000 (in Russian). The comments below are prepared for the English translation of this book.

1981-9. Consider closed contractible (bounding a disk on the universal covering) curves of constant geodesic curvature $K \neq 0$ on a surface M^2. There are at least as many such curves as critical points of a function on M^2. *Counterexample: horocycles on a surface of constant negative curvature. However, for T^2 and S^2 this conjecture has not been disproved.*

1982-24. Can the center of mass of a convex domain in a homogeneous sphere coincide with the center of the sphere? Since it cannot, it makes sense to try to prove the existence of two closed curves (magnetic trajectories) of constant positive geodesic curvature on the sphere as follows. Fiber the space of convex discs over S^2, find constrained critical points along fibers using variational methods, and then apply Morse theory techniques to look for critical points along the base.

1984-4. Prove that on T^2 there are generically at least four closed (on the universal covering) curves of constant geodesic curvature $K > 0$.

1994-14. Consider a particle in a magnetic field on a Riemannian manifold of an arbitrary dimension. The magnetic field is given by a closed two-form twisting the symplectic form of the phase space. In the case of a strong magnetic field (large curvature trajectories) apply the averaging method and, at least, formulate conjectures on topological lower bounds for +the number of periodic orbits. These conjectures should generalize the theorem on the existence of $2g + 2$ curves of large geodesic curvature on a surface of genus g.

1994-35. Find lower bounds for the number of periodic orbits of a charge in a magnetic field, where the motion of the charge is confined to a surface and the magnetic field is orthogonal to the surface. Conjecturally, on a surface of genus g, a charge should generically have at least $2g+2$ periodic orbits. From a mathematical perspective, this is a problem about closed curves with given positive geodesic curvature on the surface. When the magnetic field is sufficiently strong, the conjecture is proved, cf., Problem 1994-14.

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2. Comments

As I see it now, the work will consist of an introduction of about sixty pages, the translation proper (some two hundred printed pages) and over three hundred pages of various notes and commentaries.
– Vladimir Nabokov, Letter to Henry Allen Moe, March 1953; see [Na, p. 135].

1. The magnetic problem. First recall the Hamiltonian description of the motion of a charge in a magnetic field on a manifold. Let M be a Riemannian manifold and let σ be a closed two-form on M (the magnetic field). Consider the twisted symplectic form $\omega = \omega_0 + \pi^*\sigma$ on T^*M. Here ω_0 is the standard symplectic form on T^*M and $\pi\colon T^*M \to M$ is the natural projection. The motion of a unit charge on M in the magnetic field σ is given by the Hamiltonian flow of the standard kinetic energy $H\colon T^*M \to \mathbb{R}$. In what follows we will refer to this flow as a *twisted geodesic flow*. If M is a surface, the integral curves of the twisted geodesic flow project to the curves on M whose geodesic curvature is $k = \sigma/d\,A$, where dA is the area form. Hence, the problems in question all concern the existence of periodic orbits of twisted geodesic flows. (Note that k need not be constant.)

We emphasize that, in contrast with the geodesic flow (the case when $\sigma = 0$), the dynamics of the twisted geodesic flow on a level $\{H = c\}$ depends on c. When c is large the twisted geodesic flow is close to the geodesic flow, although not necessarily equivalent to it. On the other hand, when $c > 0$ is small the flow is in some sense "governed" by σ.

To illustrate this point, consider the twisted geodesic flow on a closed surface of constant negative curvature -1 and $\sigma = dA$, [Ar1]. For $c \in (0, 1/2)$, the flow is periodic and the period goes to infinity as $c \to (1/2)-$. In this case, all orbits are periodic and contractible in M. For $c = 1/2$, the flow is the horocycle flow which is known to have no periodic orbits, [He]. (This is a counterexample to the conjecture from Problem 1994-35, cf. Problem 1981-9 and [Gi3].) When $c > 1/2$, the flow is smoothly equivalent to the geodesic flow. In particular, every free non-trivial homotopy class of a map $S^1 \to M$ contains the projection to M of a periodic orbit and there are no periodic orbits whose projections are contractible.

Hence, when studying the existence of periodic orbits, it makes sense to treat separately the cases of high and low energy levels. For an arbitrary magnetic field σ, the existence question for periodic orbits on low energy levels is still poorly understood (see, however, [Po]) and we will focus mainly on the case where σ is symplectic. Note also that periodic orbits (contractible in M) may persist for all values of H. This is true, for example, for a flat torus and positive k, [Ar2, Ko1].

Below we focus exclusively on the symplectic geometry approach to the existence problem for periodic orbits, originating from [Ar2]. A different approach based on the Morse–Novikov theory is not discussed here; see, e.g., [GN, No, NT, Ta1, Ta2].

2. Twisted geodesic flows on surfaces. In this section we briefly list results on the existence of periodic orbits of twisted geodesic flows on surfaces with non-vanishing magnetic field. Thus, throughout this section we assume that $k \neq 0$. As have already been pointed out, for a flat torus, every level $\{H = c\}$ carries at least three periodic orbits (four, if the orbits are non-degenerate) whose projection to the torus is contractible, [Ar2, Ko1]. This result, essentially solving Problem 1984-4, was obtained by Arnold in the mid-eighties as a consequence of the Conley–Zehnder theorem, [CZ]. Since then it has served as the main motivation for applications of symplectic techniques to the study of periodic orbits of twisted geodesic flows. In a similar vein, consider an arbitrary closed orientable surface M of genus g with any metric. Then, there are at least three periodic orbits on every low energy level (two, if $M = S^2$) and at least $2g + 2$ when the orbits are non-degenerate. See [Ar2, Gi1, Gi3, Ko2] and the survey [Gi2] for further details and references. This result is a partial solution of Problem 1981-9, cf. Problems 1982-24 and 1996-18.

The dynamics on low energy levels can also be studied by using the averaging method (cf. Problem 1996-17). We refer the reader to [Ar3, BS, Ca, Li, Tr] and the references therein for a detailed discussion of this method, applications of KAM, and adiabatic invariants.

3. Twisted geodesic flows in higher dimensions. Before formulating a conjecture concerning the lower bounds for the number of periodic orbits of twisted geodesic flows in higher dimensions, let us discuss some of the relevant results. Throughout this section, (M, σ) denotes a compact symplectic manifold of dimension $2m$. Note that (M, σ) is then a symplectic submanifold of (T^*M, ω). Denote by N the vector bundle over M formed by symplectic orthogonals to M in T^*M. For every $x \in M$, we have the Hamiltonian flow of d^2H on N_x and, as a result, a fiberwise linear flow on N, called the limiting flow. The Hamiltonian flow on $\{H = \epsilon\}$ for a small $\epsilon > 0$ is close, after suitable rescaling, to the limiting flow. When the eigenvalues of $d^2H|_{N_x}$ do not bifurcate, i.e., periodic orbits of the limiting flow form smooth manifolds in $\{d^2H = \epsilon\}$, the results of the previous section extend to higher dimensions, [GK1, Ke].

Assume, for instance, that the metric is conformal to an almost-Kähler metric on (M, σ), i.e., $H(X, X) = f \cdot \sigma(X, JX)$, where f is a positive function and J is an almost complex structure compatible with σ. (This assumption implies that all orbits of the limiting flow are closed and holds automatically when M is a surface.) Then, on a low energy level, there are at least $\mathrm{CL}(M) + m$ periodic orbits, [Ke], and at least $\mathrm{SB}(M)$, if the orbits are non-degenerate, [GK1]. Here $\mathrm{CL}(M)$ stands for the cup-length of M over $\mathbb{R}$ and $\mathrm{SB}(M)$ denotes the sum of Betti numbers. Under different non-bifurcation conditions the lower bounds can be improved: for example, when at every point of M all eigenvalues are distinct, there are at least $m\mathrm{CL}(M)$ periodic orbits. The lower bounds from [Ke] are obtained using Moser's method, [Mo], which can be viewed as a higher–dimensional version of the averaging over the limiting flow (cf. Problem 1994-4).

These results lead to the conjecture that in general, for a symplectic magnetic field and low energy levels, the number of periodic orbits is no less than $\mathrm{CL}(M)+1$, or even $\mathrm{CL}(M)+m$, and $\mathrm{SB}(M)$ when the orbits are non-degenerate. Note that it is still unknown whether or not periodic orbits exist on a dense set of low energy levels. However, one can show that there are contractible (actually, "small") periodic orbits for a sequence of energy values converging to zero, [GK2].

The main difficulty which arises in showing, by symplectic topology methods, that every low energy level carries a periodic orbit lies in the fact that it is hard to find a tractable variational principle which would pick up periodic orbits on a fixed energy level of non-contact type. Clearly, the fiberwise convexity of H should be used here in an essential way; see the comments to Problem 1994-13. However, the existence of periodic orbits for a dense set of levels appears to be accessible by using symplectic or Floer homology. Finally, to obtain a lower bound on the number of periodic orbits, one needs a way to show that the action functional in question has sufficiently many critical points corresponding to geometrically distinct periodic orbits.

A different perspective on the magnetic problem in higher dimensions comes from the Weinstein–Moser theorem. Consider the following question: let W be a $2m$-dimensional symplectic manifold and let $H\colon W \to \mathbb{R}$ be a proper smooth function which has a Morse–Bott non-degenerate minimum $H = 0$ along a compact symplectic submanifold M of W. Does the Hamiltonian flow of H have a periodic orbit on every energy level $\{H = \epsilon\}$, where $\epsilon > 0$ is small? We will refer to the affirmative answer to this question as the generalized Weinstein–Moser conjecture. According to Moser and Weinstein, when $W = \mathbb{R}^{2n}$ and hence M is a point, every low energy level carries at least n periodic orbits, [Mo, We]. On the other hand, taking $W = T^*M$ and H and ω as above, we see that the magnetic problem is just a particular case of the generalized Weinstein–Moser conjecture. In fact, the results of [GK1, GK2, Ke] are all proved in the context of this conjecture and apply to a broader class of Hamiltonian systems than the motion of a charge in a magnetic field.

4. High energy levels and degenerate magnetic fields. In this section we only mention some of the relevant results.

For any metric on $M = T^n$ and any magnetic field σ, almost all (in the sense of measure theory) levels of H carry at least one periodic orbit. (See [Ji1, Ji2, GK1, Lu, Ma].) This fact is established by showing that T^*M has bounded Hofer–Zehnder capacity, [HZ]. For any weakly exact σ on a compact manifold M, there exists a sequence of positive numbers $c_k \to 0$ such that every level $\{H = c_k\}$ carries a contractible periodic orbit, [Mac2, Po]. As has been shown by J. Mather, for $M = T^2$ with any σ and a non-flat metric, there is a non-contractible periodic orbit on every high energy level; see [Gi2]. It is a simple consequence of the Viterbo theorem, [Vi], that for any metric and σ on S^2, there is a periodic orbit on every high energy level (cf. Problem 1996-18). Furthermore, the results of Bialy, [Bi], on Hopf rigidity also serve as indirect evidence of existence of contractible periodic orbits when M is a torus and $\sigma \neq 0$.

The dynamics of twisted geodesic flows is much better understood for exact magnetic fields. For example, in this case there is a periodic orbit on every high energy level, as is easy to see, e.g., from [HV]. A sharper result can be obtained by directly applying variational methods; [CIPP1]. Note in this connection that for low energy levels the action functional need not satisfy the Palais–Smale condition (see, e.g., [CIPP2]). The energy lower bound arising here is closely related to Mañe's critical value; see [CIPP1, CIPP2, Man, PP3]. For exact magnetic fields on surfaces, the sharpest results for low energy levels come from the Morse–Novikov theory, [GN, Ta1, Ta2]. Finally, we refer the reader to the forthcoming paper [CMP] for a theorem closing the gap between the dynamics on low and high energy levels for exact twisted geodesic flows on surfaces.

Topological entropy of twisted geodesic flows was studied in, e.g., [Ni1, Ni2, Mac1, PP1, PP2].

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